Portfolio Choice under Parameter Uncertainty: Bayesian Analysis and Robust Optimization Comparison

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Portfolio choice under parameter uncertainty: Bayesian analysis and robust optimization comparison

Antonio A. F. Santos∗, A. M. Monteiro† and Rui Pascoal‡

Abstract

Parameter uncertainty has been a recurrent subject treated in the financial literature. The normative portfolio selection approach considers two main kinds of decision rules: expected expected utility maximization and mean-variance criterion. Assuming that the mean-variance criterion is a good approximation to the expected utility maximization paradigm, a major factor of concern is parameter uncertainty which, when it is not taken into account, can lead to meaningless portfolios. A statistical approach, based on a Bayesian analysis, can be applied to parameter uncertainty. This can be compared with a robust optimization approach where it is assumed that the value of the unknown parameters can change within a given region. Comparisons over these two approaches are performed in this paper. We consider two measures to quantify the effects of the estimation risk, one of the measures is new and extends an existing one. The results allows us to distinguish the approaches and select the one that implies lower mean losses.

JEL Classification: C11; C13; C44; C58; C63; C87; G11; G32

keywords: portfolio choice, Bayesian statistics, robust optimization, conic programming, semidefinite programming, loss distribution

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1 Introduction

Asset allocation is a kind of decision that defines the proportion of funds to be allocated to each asset from a given amount to invest. In a problem related to the choice of a portfolio, the solution is given by a set of weights associated with each one of the assets which constitutes the decision vector. The outcome of the decision is uncertain because the returns associated with each asset are random. This is a decision problem in an uncertain environment, and a rational decision agent considers the maximization of his expected utility. In the portfolio choice approach, an approximative optimization problem can be considered. It consists in minimizing the variance of the portfolio returns subject to a given mean target return, measured by the expected return of the portfolio, or in alternative, maximizing the expected return subject to a given value for the variance. This is known as the mean-variance criterion, and has been considered extensively in the literature when we want to solve an asset allocation problem (see, Markowitz (1952, 1991); Levy and Markowitz (1979); Bawa et al. (1979); Kroll et al. (1984); Pulley (1981); Chopra and Ziemba (1993)). The decision is defined through an optimization problem but the inputs of the problem are unknown. By replacing those inputs with estimates depending on the sensitivity of the problem, important deviations can happen and the decision taken might be far from the optimal one.

We compare a more statistical approach based on a Bayesian analysis and on the use of shrinkage estimators, to a more mathematical one based on robust optimization. Portfolio choice depends on the level of risk aversion, and the results associated with the referred comparisons will also depend on those levels. We consider two measures, one from Chopra and Ziemba (1993) and the other is an extension of the measure used in the comparison of direct utility maximization versus mean-variance criterion in Kroll et al. (1984). These measures will also be used to compare sub-optimal procedures. We can find that the robust optimization constitutes a very useful tool to be included into the portfolio choice problem, and for the data and inputs used in this article, it can induce some gains when compared with more traditional approaches based on the estimation of unknown parameters. The mix of the two approaches can lead to further improvements.

The rest of the paper is organized as follows: in Section 2 we briefly discuss portfolio allocation, uncertainty and present the mean-variance criterion. Robust estimators based
on an empirical Bayesian approach are discussed in Section 3. Section 4 is devoted to robust optimization. Empirical comparisons are given in Section 5. We provide a brief conclusion in Section 6.

2 Portfolio allocation and uncertainty

Portfolio allocation constitutes a decision rule that distributes the wealth between a set of assets (stocks, bonds, derivatives, etc). The result of this decision (increase or decrease in wealth) is random, since the returns associated with different assets are also random. Here we assume the perspective of a decision agent, the investor, that takes a decision in an environment of uncertainty. Assuming that he has an utility function associated with the investment and depending on the final wealth after investment, the rational decision is defined through the maximization of the expected utility. Given an initial wealth, \( W_0 \), let us denote the decision vector by \( x \) being this constituted by the proportions to invest in each asset. The optimal value is obtained by maximizing \( EU(W) \) subject to \( x \in C(x) = \{ x : x^\top 1 = 1, x \geq 0 \} \), and \( W = W_0(1 + x^\top R) \), where \( 1 = (1, \ldots, 1)^\top \) and \( R \) is the vector of the random returns. The function \( U \) can assume different forms and depends on the characteristics of the investor, mainly it depends on his attitude towards the risk.

Due to the inherent difficulties to consider a decision criterion based on the utility function, approximate criteria were developed. In the context of portfolio allocation the most known one is the mean-variance criterion. Assuming that the expected return and the covariance matrix for \( R \) are respectively \( E(R) = \mu \), and \( Cov(R) = \Sigma \), the mean and variance of the portfolio returns are given by \( \mu_p = x^\top \mu \) and \( \sigma_p^2 = x^\top \Sigma x \), respectively.

Several versions of the mean-variance criterion have been adopted in the literature. Here, using a parameter of tolerance to risk, \( \lambda, 0 \leq \lambda \leq 1 \) representing the trade-off between risk and mean return, we adopted the version that minimizes \( \lambda x^\top \Sigma x - (1 - \lambda)x^\top \mu \), subject to \( x \in C(x) \). Following Levy and Markowitz (1979), Pulley (1981) and Kroll et al. (1984), for a given utility function, the portfolios obtained through the mean-variance criterion are very similar to the ones obtained by the direct maximization of the expected utility.

Since the benchmark is the portfolio obtained directly from the expected utility max-
mization criterion, the portfolios obtained through the mean-variance criterion must represent sub-optimal decisions. Consider the expected utility associated with the first criterion denoted by \( EU(\cdot) \) and for the second criterion \( E^*U(\cdot) \). The ratio between the two expected utilities, \( EU^*(\cdot)/EU(\cdot) \) gives an idea of the differences between the performance of both decision rules. On the other hand, Kroll et al. (1984) proposed a measure that seems to be better than just the ratio above, as it is a measure that is invariant to the different scales used in measuring the expected utilities. In this measure, it is used the expected utility associated with a portfolio with equal weights \( E_kU(\cdot) \), assuming that \( x \) has dimension \( k \) and \( x_i = 1/k \), for \( i = 1, \ldots, k \). In this case the proposed measure is given by

\[
I_{KLM} = \frac{EU^*(\cdot) - E_kU(\cdot)}{EU(\cdot) - E_kU(\cdot)}. \tag{1}
\]

The upper limit for this measure is equal to one, corresponding to the case where the mean-variance criterion gives exactly the same results compared with the direct maximization of the expected utility. Chopra and Ziemba (1993) defined the Cash Equivalent (CE), which is given by, \( CE = U^{-1}(EU(R,x)) \), and the portfolios are compared by the Cash Equivalent Loss (CEL), given by

\[
CEL = (CE_0 - CE_x)/CE_0,
\]

which is the relative loss due to choosing a suboptimal portfolio \( (x) \) instead of the optimal one \( (x_o) \). Both of these measures are used to compare the losses associated with the decision that a given decision agent defines without knowing the true value for the parameters. The CEL is used directly as it expresses already a loss function, and a redefinition of the \( I_{KLM} \) is made to be used as loss function.

2.1 The mean-variance criterion

We will follow the approach referred previously that establishes the mean-variance criterion as a good approximation. In the final of the 80s and begining of the 90s, several authors (Best and Grauer, 1991; Chopra and Ziemba, 1993) tried to explain why mean-variance criterion was not more widely used namely, why it was not used extensively by decision agents acting in the financial markets. One of the reasons was the lack of diversification of the portfolios resulting in some counterintuitive solutions. In fact, the input parameters of the subjacent optimization problem are unknown to the investor and must
be replaced by some kind of estimates. Usually the maximum likelihood estimates were
used, however, as it was shown by Best and Grauer (1991) and Chopra and Ziemba (1993),
the optimization problem is extremely sensitive to variations in the inputs, specially in
the vector of the expected value of the returns. Small variations in this vector can lead
to extreme variations in the optimal solution of the problem so, instead of using the true
optimization problem, the decision is defined through the optimization problem

$$\begin{align*}
\text{minimize} & \quad \lambda x^\top \hat{\Sigma} x - (1 - \lambda)x^\top \hat{\mu} \\
\text{subject to} & \quad x \in C(x)
\end{align*}$$

(2)

where $\hat{\Sigma}$ and $\hat{\mu}$ are the Maximum Likelihood (ML) Estimates, which replace the true value
of the parameters. As the estimates $\hat{\mu}$ and $\hat{\Sigma}$ differ from the true values, and in order
to be able to establish comparisons, we perform simulations sampling from distributions
where true values for the parameters are assumed. Since we want to compare with other
results found in literature, the parameters found in Chopra and Ziemba (1993) are the
ones used. By solving the standard quadratic optimization problem, assuming that we
know the true values, we obtain a decision vector $x^o$. On the other hand, replacing the
true values of the parameters by the estimates, the decision obtained, $x^*$, is necessarily
sub-optimal. The comparison of the two vectors $x^o$ and $x^*$ does not give the full insight of
the dimension of the sub-optimality. As in Chopra and Ziemba (1993), comparisons are
made for an investor with a specific utility function, in this case, $U(W) = -\exp(-\gamma W)$,
where $\gamma$ is the parameter of risk aversion. The decision rules are compared through the
loss of expected utility implied by the sub-optimal decision.

With the aim of comparing decision rules, we recover the measure proposed in Kroll
et al. (1984), in another context, and through its redefinition, we measure the loss of
expected utility in this context. The maximum value of (1), achieved when there is no
loss of expected utility, is equal to one. Here, we consider a loss function $L_{KLM}$ given by

$$L_{KLM}(x^o, x^*) = 1 - I_{KLM}.$$  

(3)

To keep the results comparable, we have simplified it by assuming an utility function
that depends on the return of the portfolio $U(R_p) = -\exp(-\gamma R_p)$, and like in Chopra
and Ziemba (1993), we consider the returns for monthly periods where normal distri-
bution for returns is a good approximation. The expected utility is then $EU(R_p) =
\[ -\exp\left(\frac{\gamma^2}{2}\sigma^2_p - \gamma\mu_p\right). \]

When the decision agent searches for information about the parameters in a given sample, by replacing the true values of the parameters by the sample estimates, it can happen by chance that \( x^* = x^o \), and in that case, there is no utility loss. However, the decision dependent loss function is random, and it only makes sense to evaluate its distribution. When two sub-optimal decision rules are considered, we compare the respective distributions for the loss functions.

### 3 Robust estimators

In the Statistics literature the mean-variance optimization problem, when there is uncertainty associated with the parameters, has been known as an estimation risk problem. The true values of the parameters are not known, and in the assets allocation problem two sources of risk have to be considered, one that comes from the randomness of the returns and a second that comes from the uncertainty associated with the value for the parameters.

To address this problem, in the statistical perspective, instead of just ML estimators for the parameters, it was considered the use of robust estimators which is very common in statistical problems. Sometimes they assume the form of shrinkage estimators. The shrinkage is made to a common value, establishing a trade-off between unbiasedness and efficiency of the estimators. This kind of estimators are also common when a Bayesian analysis is considered. In this context, everything that is unknown can be modeled through a probability distribution, inclusive the value of the parameters.

Bawa et al. (1979) were pioneers in dealing with estimation risk in the asset allocation setting. Other important contributions have appeared namely, Jorion (1986) and Frost and Savarino (1986). They have used the Bayesian perspective to address the problem and the most known results, the ones that seem to be the best results in a decision analysis setting, are the ones presented by Jorion (1986).

To deal with estimation risk within a Bayesian analysis, the posterior distribution for the parameters is considered, which is a combination of the likelihood function reflecting the information in the sample, and the prior distribution for the parameters. The posterior distribution characterizes the uncertainty associated with the parameters. Applying this to the asset allocation problem, the distribution of the returns is computed as the
average of the corresponding conditional distribution for the different possible values of the parameters, which gives the a-posteriori predictive distribution of the returns. The parameters (mean vector and covariance matrix) that define this distribution are going to constitute the inputs of the Mean Variance (MV) optimization problem.

Among the approaches that have been considered (Bawa et al., 1979; Frost and Savarino, 1986; Jorion, 1986), we use the methods developed by Jorion (1986). As was demonstrated by Chopra and Ziemba (1993), the main sensitivity is due to the vector of means, with variances and covariances playing a less important role. In this way, Jorion (1986) considered only the uncertainty associated with the vector of means, and has defined a prior distribution for this vector,

\[ f(\mu|\eta, \delta) \propto \exp\left(-\frac{1}{2}(\mu - \eta)^\top \delta \Sigma^{-1}(\mu - \eta)\right) \]

where \( \eta \) and \( \delta \) are the hyperparameters of the prior distribution. Since Jorion (1986) has adopted an empirical Bayes approach, those hyperparameters are estimated through a sample of dimension \( n \) such that, for the vector of returns the sample mean is \( \bar{r} \) and the sample covariance matrix is \( S \). The covariance matrix is also estimated through the sample and using \( S \). The usual unbiased sample covariance matrix is defined by

\[ \Sigma = \frac{n - 1}{n - k - 2} S, \]

where \( k \) is the number of assets to be considered. Assuming normality of the returns, the predictive distribution of the returns follows a normal distribution with mean

\[ E(R) = (1 - w)\bar{r} + w\bar{r}_0 \]

and covariance matrix

\[ \text{Cov}(R) = \left(w + \frac{n + 1}{n + \delta}\right) \Sigma + \frac{w}{n} \frac{11^\top}{1^\top \Sigma^{-1} 1}, \]

where

\[ w = \frac{\delta}{n + \delta}, \quad \bar{r}_0 = \frac{1^\top \Sigma^{-1} \bar{r}}{1^\top \Sigma^{-1} 1}. \]

The hyperparameter \( \eta \) is defined through \( \bar{r}_0 \), and as the other hyperparameter \( \delta \) defines \( w \), this parameter is approximated through the sample as

\[ \hat{w} = \frac{k + 2}{k + 2 + \Omega} \]

7
with \( \Omega = n(\bar{r} - 1r_0)^\top \Sigma^{-1}(\bar{r} - 1r_0) \).

Considering the Mean-Variance criterion like in (2), instead of using the ML estimators \( \hat{\mu} \) and \( \hat{\Sigma} \), the vector of means (4) and the covariance matrix in (5) are used instead. Given a covariance matrix for the returns, the weights associated with the minimum variance portfolio are given by \( x^m = (1\top \Sigma^{-1})/(1\top \Sigma^{-1}1) \), and \( r_0 \) is the sample mean return associated with the portfolio with minimum variance. The estimator now used for the mean of the returns is a shrinkage estimator that compresses the estimator towards the mean return of the minimum variance portfolio. The covariance matrix is also a modified version of the sample covariance matrix which accounts for the parameter uncertainty.

### 4 Robust optimization

As we pointed out before, for real world problems inputs are not always known. Our goal is to deal with uncertainty of the inputs when considering an optimization problem. This lack of information can bring extreme negative returns for the MV optimization problem. The robust optimization approach tries to prevent these extreme effects by considering the uncertainty when modeling the optimization problem, and minimizing the losses in a worst-case scenario which results in a minimax approach. Ben-Tal and Nemirovski (2000) and El Ghaoui and Lebret (1997) proposed to model unknown parameters in an optimization problem using ellipsoidal uncertainty sets. Robust factor models were presented by Goldfarb and Iyengar (2003) to define asset returns where the uncertainty was modeled by ellipsoidal sets. Worst case Value-at-Risk (VaR) with ellipsoidal sets was used to model uncertainty of \( \mu \) and \( \sigma \). Lutgens et al. (2006) extended the approach in order to include financial options. Tütüncü and Koenig (2004), Halldórsson and Tütüncü (2003) and Gregory et al. (2011) considered the robust portfolio optimization problem where the expected returns and the covariance matrix elements vary within given intervals (polyhedral sets). Lobo (2000) studied box and ellipsoidal uncertainty sets for the mean and covariance matrix. Ceria and Stubbs (2006) proposed a robust portfolio model with an uncertainty region over the expected returns. They assumed that the covariance of the returns is known exactly or that the uncertainty related to the covariance matrix is given by a finite number of discrete scenarios. Meucci (2009, 2011) and Schöttle and Werner (2009) also provided approaches for the problem. Ye et al. (2012) proposed a robust mean-variance...
portfolio problem in the form of a conic programming problem and semidefinite programming problem. The uncertainty sets are on the second moment matrix of returns and on the mean vector of returns (componentwise bounds). Chen and Kwon (2012) presented a robust portfolio selection model casted as a standard linear programming formulation.

The diversity of uncertainty sets results in significant changes when dealing with the optimization problem. In fact, according to what we have exposed before, Gregory et al. (2011) and Halldórsson and Tütüncü (2003) consider the portfolio choice within a mean-variance framework and the uncertainty set assumes the form,

\[ \mathcal{U} = \{ (\mu, \Sigma) : \mu_L < \mu < \mu_U; \Sigma_L < \Sigma < \Sigma_U, \Sigma \succeq 0 \}, \]

the inequalities are componentwise and \( \Sigma \succeq 0 \) means that \( \Sigma \) is a positive semidefinite matrix. Substituting \( \mu \) by \( \mu_L \) and \( \Sigma \) by \( \Sigma_U \), the problem is then solved as a semidefinite programming (SDP) problem. Goldfarb and Iyengar (2003) depart from polyhedral sets and propose an ellipsoidal set for the expected returns that corresponds to confidence regions for the estimates of parameters, being the covariance matrix fixed. This problem is reformulated as a Second Order Cone Programming (SOCP) problem. Ceria and Stubbs (2006) solves also an SOCP problem and Ye et al. (2012) casts the problem as an SDP problem.

### 4.1 Conic and semidefinite optimization

In order to solve the robust formulation we will need to choose carefully the uncertainty sets since the accuracy of the results will depend on it. We will consider two types of uncertainty sets: ellipsoidal for the expected returns \( \mu \) and polyhedral for the covariance matrix \( \Sigma \). We consider an ellipsoid \( E \) defined by:

\[ E = \{ \mu : \mu = Av + b \mid v^\top v \leq 1, v \in \mathbb{R}^k \}, \]

(6)

where \( A \in \mathbb{R}^{k\times k} \), \( A \) is nonsingular and \( b \in \mathbb{R}^k \). As we see, to identify an ellipsoid we only need to define the matrix \( A \) and the vector \( b \). Considering computational issues, the minimization of a linear function when the uncertainty set is an ellipsoid can be more computationally efficient when comparing to other general sets. Considering a symmetric positive semidefinite (resp. definite) matrix \( D \), the ellipsoid \( E \) can also be given by a
(resp. strictly) convex quadratic inequality

\[ E = \{ \mu : (\mu - b)^\top D(\mu - b) \leq 1 \}. \]  

(7)

In general, ellipsoid sets adjust better than polyhedral sets to the data, \( \% \alpha \) of expected returns lie within an ellipsoid defined by

\[ E = \{ \mu : n(\mu - \hat{\mu})^\top \hat{\Sigma}^{-1}(\mu - \hat{\mu}) \leq \chi^2_k(1 - \alpha) \}, \]

(8)

where \( \hat{\mu} \) is the vector of the sample mean values for the returns and \( \hat{\Sigma} \) is the sample covariance matrix of the estimates of the returns.

We now present the formulation of our problem considering the uncertainty region for the expected returns as an ellipsoidal set and for the covariance matrix as a polyhedral set, we also add the following constraint on the variances, \( \sum_{i=1}^k \sigma_{ii} = \sum_{i=1}^k \hat{\sigma}_{ii} \). The uncertainty sets are given by

\[ U_\mu = \{ \mu \in \mathbb{R}^k : (\mu - \hat{\mu})^\top \hat{\Sigma}^{-1}(\mu - \hat{\mu}) \leq \delta^2 \} \]

and

\[ U_\Sigma = \left\{ \Sigma \in S^+_k : \Sigma_l \leq \Sigma \leq \Sigma_u, \sum_{i=1}^k \sigma_{ii} = \sum_{i=1}^k \hat{\sigma}_{ii} \right\}, \]

where \( S^+_n \) is the set of positive semidefinite matrices.

Let’s consider the problem

\[ \text{minimize } \{ \max_{\mu, \Sigma} \lambda x^\top \Sigma x - (1 - \lambda)x^\top \mu \} \]

subject to \( x \in C(x) \)

\[ \mu \in U_\mu \]

\[ \Sigma \in U_\Sigma \]

Next, we will express the constraints \( \mu \in U_\mu \) and \( \Sigma \in U_\Sigma \) in an equivalent form involving, respectively, second order cone programming constraints and semidefinite constraints.

Let’s consider the constraint

\[ (\mu - \hat{\mu})^\top \hat{\Sigma}^{-1}(\mu - \hat{\mu}) \leq \delta^2 \iff \]

\[ \mu^\top \hat{\Sigma}^{-1} \mu - 2\hat{\mu}^\top \hat{\Sigma}^{-1} \mu + \hat{\mu}^\top \hat{\Sigma}^{-1} \hat{\mu} - \delta^2 \leq 0, \]
decomposing $\hat{\Sigma}^{-1}$ as $P^\top P$ and choosing $t$ such that $t \geq \mu^\top \hat{\Sigma}^{-1} \mu$, we define a new variable $y$ such that $y = P\mu$ and consequently:

$$t \geq \|y\|^2_2.$$  

Thus, the constraint (9) can be substituted by

$$t - 2\hat{\mu}^\top \hat{\Sigma}^{-1} \mu + \hat{\mu}^\top \hat{\Sigma}^{-1} \hat{\mu} - \delta^2 = 0$$

$$P\mu = y$$

$$t \geq \|y\|^2_2;$$

these expressions assure that $(t, \frac{1}{2}, y)$ belongs to a rotated quadratic cone $Q_{r}^{k+2} = \{(t, \frac{1}{2}, y) \in \mathbb{R}^{k+2} : t \geq \sum_{i=1}^{n} y_i^2 \}$. So, our problem can be casted as

$$\min_x \{ \max_{\mu, \Sigma} \lambda x^\top \Sigma x - (1 - \lambda) x^\top \mu \}$$

subject to

$$x \in C(x)$$

$$t - 2\hat{\mu}^\top \hat{\Sigma}^{-1} \mu + \hat{\mu}^\top \hat{\Sigma}^{-1} \hat{\mu} - \delta^2 = 0$$

$$P\mu - y = 0$$

$$(t, \frac{1}{2}, y) \in Q_{r}^{k+2}$$

$$\Sigma \in \mathcal{U}_r.$$

The inner maximization problem can be considered as the composition of two problems: an SOCP problem related to the variable $\mu$ and an SDP problem related to the variable $\Sigma$. In fact, since $\Sigma$ is a positive semidefinite matrix it belongs to the cone of positive semidefinite matrices, $S_+$, consequently the second problem is an SDP problem, which is a generalization of a cone programming problem. The problems are stated, respectively, as

$$\max_{\mu} -\mu^\top x$$

subject to

$$t - 2\hat{\mu}^\top \hat{\Sigma}^{-1} \mu + \hat{\mu}^\top \hat{\Sigma}^{-1} \hat{\mu} - \delta^2 = 0$$

$$P\mu - y = 0$$

$$(t, \frac{1}{2}, y) \in Q_{r}^{k+2}$$

(10)
and

\[
\begin{align*}
\text{maximize} & \quad \Sigma x^\top \Sigma x \\
\text{subject to} & \quad \Sigma \in \mathcal{U}_\Sigma.
\end{align*}
\] (11)

Both problems are convex since the objective functions are convex and we have linear equality and inequality constraints, second order cone constraints and positive semidefinite constraints.

When comparing the approach related solely to the uncertainty on the expected returns $\mu$, with the approach related to jointly uncertainty sets in $\mu$ and $\Sigma$, we confirm the idea that the uncertainty related to the vector of expected returns is the most determinant issue in the robust portfolio approach.

5 Empirical comparisons

In the empirical section of this paper, we use the vector of the means and the covariance matrix presented in Chopra and Ziemba (1993) as the true values for the parameters. Considering these true values for the inputs, and $\lambda$ such that $0 \leq \lambda \leq 1$, we computed the optimal portfolios which, in the space formed by $(\sigma_p, \mu_p)$, belong to the set of Pareto optimal solutions, also known as the efficient frontier.

As a first approach to the problem we computed the optimal portfolio for the investor with a tolerance to risk of $t = 25$. It is identified as the stardot in the efficient frontier depicted in Figure 1. The optimal portfolios are given in the Table 1.

To give an idea of the sensitivity of the optimization problem to changes in the inputs, assuming the true values of the parameters just referred, and using a normal distribution, a simulation was performed. We considered samples of 48 observations (4 years of monthly returns) as the inputs for the classical ML approach. One hundred samples were generated and the respective optimization problems were solved. Relatively to the true value of the parameters, the expected value of the portfolio returns and respective standard deviation were calculated. The points are plotted as the plus signals in Figure 1, they are widely spread over the space $(\sigma_p, \mu_p)$, sometimes very far from the optimal trade-off between risk and mean return. To be able to visualize better the difference between the optimal portfolio and the ones that are suboptimal, we have excluded from the plot the points that were out of the range of standard deviation and expected return defined by the efficient
frontier. However, they were many, which reinforces the idea of the extreme sensitivity of the problem to errors in the inputs.

To make the results comparable with the ones from Chopra and Ziemba (1993) we used the same values for the parameter of risk tolerance $t$: $t = 25$, $t = 50$ and $t = 75$. These values correspond to $\gamma = 0.08$, $\gamma = 0.04$ and $\gamma = 0.027$. In our optimization problem we obtain the corresponding values of $\lambda$: $\lambda = 0.0385$, $\lambda = 0.0196$ and $\lambda = 0.0132$. In Table 1 we present the values for the means and variances associated with the ten assets considered. We present also the computed optimal portfolios associated with each risk tolerance parameter value. Recognizing that for $t = 50$ and $t = 75$ the level of diversification is already low, we added another risk tolerance parameter value $t = 10$.

![Figure 1: This figure depicts the efficient frontier, and identifies the optimal portfolio for the agent with $t = 25$ (stardot in the efficient frontier), which comes from the solution of problem (2).](image)

The optimal portfolio for each level of tolerance to risk are plotted in the Table 1. Our aim is to measure the distances between these optimal portfolios and the ones that
Table 1: Inputs taken from Chopra and Ziemba (1993) (just means and variances are shown but the entire covariance matrix was considered), and optimal portfolios computed for different levels of tolerance to risk.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Var.</th>
<th>$x_{t=10}$</th>
<th>$x_{t=25}$</th>
<th>$x_{t=50}$</th>
<th>$x_{t=75}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alcoa</td>
<td>1.561</td>
<td>77.983</td>
<td>0.0096</td>
<td>0.0876</td>
<td>0.0351</td>
<td>0.0000</td>
</tr>
<tr>
<td>Amex</td>
<td>1.948</td>
<td>71.546</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0082</td>
<td>0.0006</td>
</tr>
<tr>
<td>Boeing</td>
<td>1.907</td>
<td>100.802</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1055</td>
</tr>
<tr>
<td>Chevron</td>
<td>1.580</td>
<td>74.330</td>
<td>0.1848</td>
<td>0.1871</td>
<td>0.1626</td>
<td>0.8939</td>
</tr>
<tr>
<td>Coke</td>
<td>2.164</td>
<td>35.863</td>
<td>0.4235</td>
<td>0.6018</td>
<td>0.7940</td>
<td>0.0000</td>
</tr>
<tr>
<td>Du Pont</td>
<td>1.601</td>
<td>47.289</td>
<td>0.0074</td>
<td>0.0158</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>MMM</td>
<td>1.489</td>
<td>33.829</td>
<td>0.1031</td>
<td>0.0014</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>P &amp; G</td>
<td>1.625</td>
<td>31.793</td>
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result from approximative procedures, where parameters are replaced by ML estimates and robust estimates. The novelty of this paper is the comparison between the robust statistics approach, and a more recently one found in the literature: the robust optimization approach. Since the investor does not know the true value for the parameters he always defines a sub-optimal decision. We are going to compare sub-optimal decisions and try to establish if there are outputs that allow to obtain decisions near the optimal one.

The MV criterion allows different approaches as we have seen before: MLE, robust statistics and robust optimization. The robust optimization approach is not a “pure” one in the sense that it also uses the information in the sample to define the structure of the problem, namely, the definition of the uncertainty set. In this sense, for each situation, the decision agent uses information in the sample to define his decision. As the sample is random, different information is obtained which originates random decisions and random losses. We compare the differences between the resulting expected utilities: the optimal
one from the true value for the parameters, and the other that is obtained from the sub-optimal portfolio.

The investor does not know the true values for the inputs. If sample information is used to define the inputs in the classical, robust statistics or robust optimization approach, we obtain a given portfolio which is necessarily sub-optimal. The loss of expected utility can be measured, however for different samples we obtain different values for the losses. In a wider sense, we compare the probability distributions of the losses associated with each decision rule.

To make our results more robust to the choice of the loss function, we used the CEL measure already proposed by Chopra and Ziemba (1993) and a redefinition of the measure proposed by Kroll et al. (1984), the $L_{KLM}$ (3). The results obtained are robust to the choice of the loss functions in the sense that the same ranking for different decisions is obtained using both kind of loss functions.

To illustrate our results we approximated numerically, trough simulations, the distribution functions for each loss function associated with each decision rule. For illustrative proposes, we considered a level of tolerance to risk of $t = 25$, $n = 48$ sample sizes, $\delta = 1.65$ as the parameter for the uncertainty set for the mean, and for the polyhedral set associated with the covariance matrix a shift of -5 and 5 for each element of the matrix. Other simulations were performed and the code in Matlab that uses the Mosek optimization package is available upon request.

The graphical representation of the shape of the distribution functions is depicted in Figure 2, which refers only to the loss function $L_{KLM}$. In Table 2, for each decision rule, the common mean and standard deviation measures were computed for the two measures (loss functions) with the parameters just referred above. The graphic shows that the robust approaches (statistics and optimization) dominate the classical one (MLE). This is not new as this was already establish in the literature. In fact, better decisions are obtained when the estimation risk associated with the parameters are incorporated in the decision (Bawa et al., 1979; Frost and Savarino, 1986; Jorion, 1986). The new results in this paper are related to the comparison with the robust optimization approach. When the uncertainty set to the vector of means is considered, there seems to be an improvement in the outcome of the decision rules, since the distribution function moves a little further to the left, obtaining lower values for the mean losses and also lower variances. However,
there is a different behavior between robust statistics and robust optimization: at high levels of \( L_{KLM} \) robust optimization is better, and at low levels of \( L_{KLM} \) this difference becomes less significative.

Our results seem to be even more robust when we compare the decision rules that use robust optimization with uncertainty sets with, and without, the component associated with the covariance matrix. Using the comparison of the distribution functions and the respective statistics, mean and standard deviations, there is no clear cut distinction. The inclusion on the uncertainty set of the components associated with the covariance matrix does not give substantial improvements in the decision rules. This is in line with previous results which state that the main sensitivity of the decision problem is to the vector of the means, and not so much to the variances and covariances.

The results presented are valid in the context described in this article, but in most aspects are comparable with the ones found in the literature. In future work more general results can be obtained by redefining the optimization problems considering larger dimensions (greater number of assets), different type of constraints, liquidity, upper limits to the weights or even transaction costs. Well calibrated algorithms and robust software have to be developed to accomplish this goal.

\[
\begin{array}{l|cc|cc}
\hline
\text{Decision rule} & \multicolumn{2}{c|}{L_{KLM}} & \multicolumn{2}{c}{\text{CEL}} \\
 & \text{Mean} & \text{Std} & \text{Mean} & \text{Std} \\
\hline
\text{MLE} & 0.7973 & 0.6397 & 0.3864 & 0.3204 \\
\text{R Stat} & 0.4021 & 0.2941 & 0.1902 & 0.1375 \\
\text{R Opt Mean} & 0.3511 & 0.2297 & 0.1703 & 0.1172 \\
\text{R Opt M-Var} & 0.3641 & 0.2265 & 0.1693 & 0.1085 \\
\hline
\end{array}
\]

Table 2: Main statistics, mean and standard deviation characterizing the distribution of the loss functions associated with each decision rule.
Figure 2: Probability distribution of the loss function $L_{KLM}$, for four decision rules, MLE (parameters replaced by their sample estimates), Robust Statistics (R Stat), Robust Optimization with an uncertainty set (ellipsoidal) for the vector of means (R Opt Mean), and Robust Optimization with two uncertainty sets (R Opt M-Var), one for the vector of means (ellipsoidal) and the other for the covariance matrix (polyhedral).
6 Conclusions

Our results are in line with what is commonly presented in the literature. The optimization problem associated with the asset allocation is extremely sensitive to changes in the inputs, specially in the vector of the mean returns. Diverse strategies can be devised to make the problem less sensitive, obtaining more stable decisions. Two strategies were compared in the optic of a specific investor. Considering that a truly optimal decision (asset allocation) exists, the decision agent is going to use the information available in a given sample to define its decision. This information can be used in a more direct way considering the use of robust statistics, where the estimated values for the parameters are used in the optimization problem. This information can also be used in a more indirect manner, where the sample information can be used to define the uncertainty sets in the robust optimization problem. The results show that there are improvements in the decision process, and the ones coming from the robust statistics and the robust optimization dominate the one that just replaces the true values of the parameters by sample ML estimates. The results may vary from investor to investor, depending on their attitude towards risk, the characteristics of their utility functions, the distribution used as the true distribution of the returns, the sample size and the definition of the uncertainty sets in the robust optimization problem. However, our results are similar to the ones in the literature and present the novelty of measuring the loss for a specific investor. On the other hand, when we apply the robust optimization with an uncertainty set associated to the covariance matrix, using semidefinite programming, the results obtained do not differ much from the ones when the uncertainty is only considered for the vector of mean returns, which illustrates the less sensitiveness of the problem to the inputs in the covariance matrix, as it was already found in the literature.

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