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Efficient Skewness/Semivariance Portfolios

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Efficient Skewness/Semivariance Portfolios

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Abstract

This paper proposes a new way to measure and deal with risk within the portfolio selection problem using a skewness/semivariance biobjective optimization framework. The solutions of this biobjective optimization problem allow the investor to analyse the efficient trade-off between skewness and semivariance. Due to the endogeneity of the cosemivariance matrix, the biobjective problem is solved using a derivative-free algorithm based on direct multisearch. For four datasets, collected from the Fama/French benchmark collection, the direct multisearch was able to determine the in-sample Pareto frontier. The out-of-sample performance of the skewness/semivariance model was assessed by choosing three portfolios (the portfolio that maximizes a skewness per semivariance ratio, the portfolio that maximizes the Sharpe ratio and the portfolio that maximizes the Sortino ratio) at each in-sample Pareto frontier and measuring their performance in terms of skewness per semivariance ratio, Sharpe ratio, Sortino ratio and turnover. The results show that the efficient skewness/semivariance portfolios are consistently competitive, and often superior, comparatively to the benchmark portfolios considered. Both in-sample and the out-of-sample performance analysis were conducted using three different benchmark returns for the semivariance computations.

JEL Classification: C44; C58; C61; C63; C88; G11

Keywords: portfolio selection, semivariance, skewness, multiobjective optimization, derivative-free optimization, direct multisearch

1 Introduction

The Nobel laureate Henry Max Markowitz noted that the investor who wishes to allocate his wealth to a set of securities must keep in mind not only the maximization of profit associated with this allocation, but also the minimization of the risk involved. In his ground-breaking work, Markowitz (1952) proposes a mean-variance optimization model intended to minimize the portfolio risk (measured by its variance) for a given level of expected return, over a set of feasible portfolios. By varying the level of expected return, the model determines the efficient frontier, as the set of mean-variance efficient portfolios.

The theoretical consistency of the mean-variance model proposed by Markowitz (1952) with the Von Neumann-Morgenstern axioms of choice (Von Neumann and Morgenstern, 1953) is dependent on either the security returns following a normal distribution or a quadratic utility

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function. However, there is some misunderstanding about this issue. Occasionally it is said that the application of the mean-variance analysis assumes normal return distributions or quadratic utility functions. This is a mistake which was recently discussed and clarified by Markowitz (2014) himself. Normal return distributions or quadratic utility functions are sufficient but not necessary conditions for the use of mean-variance analysis. A careful choice on a mean-variance efficient frontier can approximately maximize expected utility for a variety of utility functions. Nevertheless, if one knows that the distribution of returns is not normal and that the utility function can take other forms than the quadratic one, another type of analysis may be more suitable.

Several studies, such as Mandelbrot (1963), Fama (1965), Beedles (1979), Campbell and Hentschel (1992), and Turner and Weigel (1992) showed that portfolio returns are not, in general, normally distributed. On the other hand, studies such as Rubinstein (1973), Arditti (1975), Kraus and Litzenberger (1976), Kane (1982), and Harvey and Siddique (2000) found that investors have preferences for skewness, suggesting that usually utility functions are not quadratic. In fact, recent studies such as Lai (1991), Chunhachinda et al. (1997), Athayde and Flôres (2004), Mencia and Sentana (2009), Harvey et al. (2010) suggest gains from taking into account higher moments in portfolio selection.

With non-normal return distributions, the use of a downside risk measure is more suitable than the traditional use of the variance (Nawrocki, 1999). Roy (1952) was the first to observe the importance of a downside risk measure in portfolio selection, in the form of a "safety first" rule that measures the probability of outcomes falling below a predetermined target return (Sing and Ong, 2000). Markowitz (1959, 1991) recognized the importance of the Roy's work, arguing that there are more plausible measures of risk than the variance, and proposes the use of a below-mean semivariance or a target-return semivariance. These metrics belong to a more general family of downside risk measures known as lower partial moments (Bawa, 1975, Fishburn, 1977). Quirk and Saposnik (1962) confirmed the theoretical superiority of the semivariance versus the variance, while Ang and Chua (1979) showed the superiority of the target-return semivariance relative to the below-mean semivariance. Ang and Chua (1979) also showed that only the target-return semivariance is a measure of risk in accordance with the Von Neumann-Morgenstern axioms of choice. The difficulties in computing the semivariance, as originally proposed by Markowitz, results from the endogeneity of the cosemivariance.

In this paper, we suggest a direct analysis of the efficient trade-off between skewness and semivariance by means of a skewness/semivariance biobjective optimization problem. We deal with skewness as a third moment tensor as in Athayde and Flôres (2004) and consider an endogenous cosemivariance matrix as in Markowitz (1959). Due to the endogeneity of the cosemivariance matrix, a derivative-free algorithm (based on direct multisearch) is used to obtain the solution of the biobjective optimization problem. Direct multisearch is a class of methods for the solution of multiobjective optimization problems that does not use derivatives and does not aggregate or scalarize any of the objective function components. It essentially generalizes all direct-search methods of directional type from single to multiobjective optimization. For a complete description of direct multisearch see the algorithmic framework in Custódio et al. (2011).

The empirical work is conducted on four datasets collected from the Fama/French benchmark collection (formed according distinct characteristics). Firstly, the Pareto frontiers on the skewness/semivariance space are computed using three different benchmark returns, corresponding to the returns of the maximum Sharpe ratio, the minimum variance and the $1/N$ portfolios, respectively. Then, an extensive out-of-sample performance analysis is implemented on three efficient skewness/semivariance portfolios from the in-sample Pareto frontiers: the portfolios with the maximum skewness per semivariance ratio, the maximum Sharpe ratio and the maximum Sortino ratio. The out-of-sample performance is measured in terms of skewness per semivariance ratio, Sharpe ratio, Sortino ratio and turnover. The efficient portfolios exhibited a competitive, and often superior, out-of-sample performance compared to the three benchmark portfolios considered: the maximum Sharpe ratio portfolio, the minimum variance portfolio and the $1/N$ portfolio. Although the efficient skewness/semivariance portfolios present a higher value of turnover, they have a consistently good performance in terms of the skewness per semivariance ratio and the Sortino ratio measures, which suggests the robustness of the efficiency provided by the skewness/semivariance model. One interesting result is that in the four datasets the efficient skewness/semivariance portfolios consistently outperform the $1/N$ portfolio in terms of Sharpe ratio, which is known to be difficult to achieve (DeMiguel et al., 2009).

The rest of the paper is organized as follows. Section 2 introduces some concepts and notions used thereafter. Section 3 describes the benchmark portfolios (the maximum Sharpe ratio portfolio, the minimum variance portfolio and the $1/N$ portfolio). In Section 4 it is shown that the expected utility of a risk averse investor is an increasing function of the skewness and a decreasing function of the semivariance. Section 5 presents the skewness/semivariance biobjective model. Some basic material about direct multisearch for multiobjective optimization is described in Appendix B. Section 6 shows the empirical results and, finally, Section 7 summarizes the main findings and discusses future research.

2 Notation

Suppose that an investor has a certain wealth to invest in a set of N securities. The return of each security i is described by a random variable R_i. Based on historical data, a return portfolio matrix, R, can be defined as

$$R = [\ R_1 \mid R_2 \mid \ \dots \ \mid R_N \ ] = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,N} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ r_{T-1,1} & r_{T-1,2} & \cdots & r_{T-1,N} \end{bmatrix},$$

where T corresponds to the total number of price observations. Let w_i, $i = 1, \dots, N$, represent the proportions of the total investment allocated to each security. Thus a portfolio can be defined by an $N \times 1$ vector w of weights which has to satisfy the constraint

$$\sum_{i=1}^{N} w_i \ = \ e^\top w \ = \ 1,$$

where e is the $N \times 1$ vector of ones. Lower bounds on the variables, of the form $w_i \geq 0$, $i = 1, \dots, n$, can be also considered if short selling is undesirable. The expected return of the security i is given by $\mu_i = E(R_i)$, for $i = 1, \dots, N$. Let $m = [\ \mu_1 \ \mu_2 \ \dots \ \mu_N \ ]^\top$ be the vector of the expected returns. Thus, the portfolio expected return can be written as

$$\mu(w) \ = \ E(R(w)) \ = \ E\left(\sum_{i=1}^{N} w_i R_i\right) \ = \ \sum_{i=1}^{N} w_i \mu_i \ = \ m^\top w.$$

The portfolio variance, is given by

$$\nu(w) = E\left[(R(w) - \mu(w))^2\right] = w^\top M_2 w,$$

where M_2 is the covariance matrix in which each entry $c_{ij} = COV(R_i, R_j)$ is computed as

$$c_{ij} = \frac{1}{T-1} \sum_{i,j=1}^{N} \sum_{t=1}^{T-1} (r_{t,i} - \mu_i)(r_{t,j} - \mu_j).$$

M_2 is symmetric and positive semi-definite.

3 Benchmark portfolios

3.1 The maximum Sharpe ratio portfolio

The Markowitz's model (Markowitz, 1952, 1959) is based on the formulation of a mean-variance optimization problem. The solution of that problem is the portfolio of minimum variance for an expected return not below a certain target value r. Therefore, the aim is to minimize the risk for a given level of return. This problem can be defined as:

$$\begin{aligned} \min_{w \in \mathbb{R}^N} \quad & w^\top M_2 w \\ \text{subject to} \quad & m^\top w \geq r, \\ & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{1}$$

Problem (1) is a convex quadratic programming problem (QP), for which the first order necessary conditions are also sufficient for (global) optimality. See Steinbach (2001), Cornnuejols and Tütüncü (2007) for a survey on portfolio optimization.

The classical Markowitz mean-variance model can be reformulated as a biobjective problem which consists of simultaneously minimizing the portfolio risk (variance) and maximizing the portfolio profit (expected return)

$$\begin{aligned} \min_{w \in \mathbb{R}^N} \quad & \nu(w) = w^\top M_2 w \\ \max_{w \in \mathbb{R}^N} \quad & \mu(w) = m^\top w \\ \text{subject to} \quad & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{2}$$

It is easy to prove that a solution of problem (1) is nondominated, efficient or Pareto optimal for problem (2). Efficient portfolios are thus the ones which have the minimum variance for at least a certain expected return, or, alternatively, those that have the maximal expected return up to a certain variance. The efficient frontier (or Pareto frontier) is typically represented as a 2-dimensional curve in the mean-standard deviation space.

In the efficient frontier the portfolio that maximizes the risk premium per risk unit deserves a special attention; i.e., the portfolio that maximizes the ratio between the reward and the variability of the investment (reward-to-variability ratio). This portfolio is obtained by maximizing the so-called Sharpe ratio

$$
\begin{aligned}
\max_{w \in \mathbb{R}^N} \quad & \frac{m^\top w - r_f}{\sqrt{w^\top M_2 w}} \\
\text{subject to} \quad & e^\top w = 1, \\
& w_i \geq 0, \quad i = 1, \dots, N,
\end{aligned}
\tag{3}
$$

where r_f is the risk-free rate.

Proposition 3.1 *The maximum Sharpe ratio portfolio (ms portfolio), solution of (3), can be found by setting $w^* = \tau^* x^*$, where (x^*, τ^*) is the solution of the following convex QP*

$$
\begin{aligned}
\min_{x \in \mathbb{R}^N, \tau \in \mathbb{R}^+} \quad & x^\top M_2 x \\
\textit{subject to} \quad & (m - r_f e)^\top x = 1, \\
& e^\top x = \tau, \\
& \tau > 0, \\
& x_i \geq 0, \quad i = 1, \dots, N.
\end{aligned}
\tag{4}
$$

3.2 The minimum variance portfolio

The minimum variance portfolio (*mv* portfolio) corresponds to the solution of the following convex QP

$$
\begin{aligned}
\min_{w \in \mathbb{R}^N} \quad & w^\top M_2 w \\
\text{subject to} \quad & e^\top w \;=\; 1, \\
& w_i \geq 0, \quad i = 1, \dots, N.
\end{aligned}
\tag{5}
$$

Regarding problem (5), it has been shown in the literature that the estimates of the mean returns are so noisy that it is completely preferable to ignore these estimates and use only the covariance matrix (Jagannathan and Ma, 2003). It is also known that not allowing for short selling on the minimum-variance portfolio has a regularizing effect (Jagannathan and Ma, 2003).

3.3 The $1/N$ portfolio

The $1/N$ rule or naive strategy ($1/N$ portfolio) is the one in which the available investor's wealth is divided equally among the available securities

$$
w_i \;=\; \frac{1}{N}, \quad i = 1, \dots, N. \tag{6}
$$

This strategy has diversification as its main goal, it does not involve optimization, and it completely ignores the data. Although a number of theoretical models have been developed in recent years, many investors pursuing diversification revert to the use of the naive strategy to allocate their wealth (Benartzi and Thaler, 2001). DeMiguel et al. (2009) evaluated fourteen models across seven empirical data sets and showed that none is consistently better than the naive strategy. A possible explanation for this phenomenon lies in the fact that the naive strategy does not involve estimation and promotes 'optimal' diversification. The naive strategy is therefore an excellent benchmarking strategy.

4 Investor expected utility

4.1 Expected utility based on skewness

Let $u(.)$ be the utility function of a typical investor. Considering the Taylor expansion of $u(R(w))$ around $\mu(w)$, then

$$\begin{aligned} u(R(w)) &= u(\mu(w)) + u'(\mu(w))(R(w)-\mu(w)) \\ &+ \frac{1}{2!}u''(\mu(w))(R(w)-\mu(w))^2 + \frac{1}{3!}u'''(\mu(w))(R(w)-\mu(w))^3 \\ &+ \sum_{j=4}^{\infty}\frac{1}{j!}u^j(\mu(w))(R(w)-\mu(w))^j. \end{aligned}$$

Applying the mathematical expectation operator, E,

$$E\left[u(R(w))\right] = u(\mu(w)) + \frac{1}{2!}u''(\mu(w))\nu(w) + \frac{1}{3!}u'''(\mu(w))\kappa(w) + \sum_{j=4}^{\infty}\frac{1}{j!}u^j(\mu(w))m^j(w),$$

where $\nu(w) = E\left[(R(w)-\mu(w))^2\right]$, $\kappa(w) = E\left[(R(w)-\mu(w))^3\right]$, $m^j(w) = E\left[(R(w)-\mu(w))^j\right]$ are the second, third and j-th (with $j = 4, 5, \ldots$) central moments of $R(w)$, respectively. Considering the third-order approximation of the expected utility, as in Joro and Na (2006), then

$$E\left[u(R(w))\right] \approx u(\mu(w)) + \frac{1}{2}u''(\mu(w))\nu(w) + \frac{1}{6}u'''(\mu(w))\kappa(w). \tag{7}$$

Since $u'''(\mu(w)) >= 0$ (see Arditti (1967) and Kraus and Litzenberger (1976)), the expected utility, $E\left[u(R(w))\right]$, of a risk averse investor is an increasing function of skewness, $\kappa(w)$, which is consistent with the desirable properties for an investor's utility function ($u'(.) > 0$, $u''(.) < 0$ and $u'''(.) > 0$) suggested by Arrow (1971).

There is an intuitive explanation of why the skewness is important for the investor. Clearly, the investor has a preference for positive skewness in order to have higher probability for extreme profit values and limited loss. Alderfer and Bierman (1970) showed empirically that investors prefer positive skewness, even if this positive skewness is associated with a lower expected return, that is, investors are willing-to-pay for positive skewness.

Arditti (1967, 1971), Rubinstein (1973), Kraus and Litzenberger (1976) and Harvey and Siddique (2000) showed theoretically that investors prefer positive skewness. Moreno and Rodríguez (2009) showed that the coskewness is taken into account by funds managers, representing an important factor in the securities selection. However, skewness is often neglected in the performance evaluation literature, possibly due to computational difficulties (Joro and Na, 2006).

4.2 Expected utility based on semivariance

If instead the variance, one considers the semivariance

$$\Sigma_B(w) = E\left\{\left[\min(R(w)-B, 0)\right]^2\right\},$$

where B represents a benchmark return and should be independent of the probability distribution being ranked (Ang and Chua, 1979). Then the utility function should have a kink at the reference point B. Based on Koekebakker and Zakamouline (2008), the utility function has the form

$$u(R(w)) = \begin{cases} u_+(R(w)) & \text{if } R(w) \geq B, \\ u_-(R(w)) & \text{if } R(w) < B, \end{cases}$$

where $u_+(.)$ is the utility function for gains and $u_-(.)$ is the utility function for losses. Considering the second-order Taylor expansion approximation of $u(R(w))$ around B, then

$$u(R(w)) \approx \begin{cases} u_+(B) + u'_+(B)(R(w) - B) + \frac{1}{2}u''_+(B)(R(w) - B)^2 & \text{if } R(w) \geq B, \\ u_-(B) + u'_-(B)(R(w) - B) + \frac{1}{2}u''_-(B)(R(w) - B)^2 & \text{if } R(w) < B. \end{cases}$$

Applying the mathematical expectation operator, E,

$$E[u(R(w))] \approx \begin{cases} u_+(B) + u'_+(B)E[(R(w) - B)] + \frac{1}{2}u''_+(B)E[(R(w) - B)^2] & \text{if } R(w) \geq B, \\ u_-(B) + u'_-(B)E[(R(w) - B)] + \frac{1}{2}u''_-(B)E[(R(w) - B)^2] & \text{if } R(w) < B. \end{cases}$$

If the investor is risk averse in the domain of losses (the utility function for losses is concave, $u''_-(.) < 0$), then the expected utility, $E[u(R(w))]$, is a decreasing function of semivariance, $\Sigma_B(w)$. Again, this is consistent with the desirable properties for an investor's utility function ($u'(.) > 0$, $u''(.) < 0$ and $u'''(.) > 0$) (Arrow, 1971).

5 The skewness/semivariance biobjective model

To overcome the limitation of the mean-variance models, some researchers used downside risk measures, which only gauge the negative deviations from a reference return level. One famous downside risk measure was introduced in the safety first criterion (Roy, 1952). Other downside risk measures were proposed in Bawa (1975), Fishburn (1977), Lee and Rao (1988), Harlow and Rao (1989), Homaifar and Graddy (1990), Harlow (1991), Nawrocki (1991), Chow and Denning (1994), Merriken (1994), Rom and Ferguson (1994). See Belzer (1994), Nawrocki (1999), Estrada (2006) for a survey on downside risk measures.

Markowitz (1959, 1991) favored one of the best-known downside risk measures: the semivariance of returns. The semivariance can be handled by considering an asymmetric cosemivariance matrix (Hogan and Warren, 1974) or considering a symmetric and exogenous cosemivariance matrix (Estrada, 2008). Another way of handling the semivariance is outside the stochastic environment, considering the fuzzy set environment as in Huang (2008).

Following Markowitz (1959), the endogenous cosemivariance matrix is the approach adopted here. Therefore the exact estimation of the semivariance of a portfolio is obtained as

$$\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cs_{ij} = w^\top M_2^-(w)w,$$

where $M_2^-(w)$ is the cosemivariance matrix in which each entry cs_{ij} is given by

$$cs_{ij} = \frac{1}{T-1} \sum_{k \in U} (r_{k,i} - B)(r_{k,j} - B), \tag{8}$$

with

$$U = \{t \mid r_{t,p} < B\} \subseteq \{1, \ldots, T-1\},$$

where $r_{t,p}$ is the portfolio return at time t. In this case the cosemivariance matrix is endogenous in the sense that a change in the portfolio's weights affects the periods when the portfolio underperforms the benchmark, which in turn affects the elements of the cosemivariance matrix (Estrada, 2008).

Skewness can be computed as a third moment tensor and can be visualized as a $N \times N \times N$ cube in the three-dimensional space. It is possible to transform the skewness tensor into a $N \times N^2$ matrix by slicing each $(N \times N)$ layer and pasting them, in the same order, sideways (Athayde and Flôres, 2004). Following this idea the skewness of a portfolio can be computed as

$$\kappa(w) = E\left[(R(w) - \mu(w))^3\right] = w^\top M_3(w \otimes w),$$

where $\otimes$ denotes the Kronecker product and M_3 is the coskewness matrix. The coskewness matrix of dimension $N \times N^2$ can be represented by N A_{ijk} matrixes of dimensions $N \times N$ such that

$$M_3 = [\ A_{1jk}\ A_{2jk}\ \ldots\ A_{Njk}\ ],$$

where $j, k = 1, \ldots, N$. The individual elements of the coskewness matrix can by obtained as

$$a_{ijk} = \frac{1}{T-1} \sum_{i,j,k=1}^{N} \sum_{t=1}^{T-1} (r_{t,i} - \mu_i)(r_{t,j} - \mu_j)(r_{t,k} - \mu_k).$$

According with Section 4, we propose the simultaneous consideration of the two investor's objectives

- maximizing the skewness $\kappa(w) = w^\top M_3(w \otimes w)$,
- minimizing the semivariance $\Sigma_B(w) = w^\top M_2^-(w)w$,

over the set of feasible portfolios. In this case, the skewness/semivariance biobjective optimization model can be written as

$$\begin{aligned} \max_{w \in \mathbb{R}^N} \quad & \kappa(w) = w^\top M_3(w \otimes w) \\ \min_{w \in \mathbb{R}^N} \quad & \Sigma_B(w) = w^\top M_2^-(w)w \\ \text{subject to} \quad & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{9}$$

By solving (9), we identify a skewness/semivariance Pareto frontier. A portfolio in this frontier is such that there exists no other feasible one which simultaneously presents a higher skewness and a lower semivariance. Given such an efficient frontier and a semivariance target, an investor may directly find the answers to the questions of what is the optimal (higher) skewness level that can be chosen and what are the portfolios leading to such a skewness level. The biojective problem (9) has two objective functions, a linear constraint and N inequality constraints. The first objective, $\kappa(w) = w^\top M_3(w \otimes w)$, is nonlinear but smooth. However, the second objective, $\Sigma_B(w) = w^\top M_2^-(w)w$, is nonlinear and nonsmooth due to the endogeneity problem on the cosemivariance matrix $M_2^-(w)$. We have thus decided to solve the problem through a derivative-free solver, based on direct multisearch (see Appendix B for a brief description of direct multisearch). This derivative-free solver was previously and for the first time used in the portfolio selection framework for solving a cardinality constrained problem (Brito and Vicente, 2014).

6 Empirical analysis

The empirical analysis is conducted on four datasets collected from the Fama/French benchmark collection, which is publicly available from the site: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. Each dataset is constructed according to different criteria (see Table 1 for a brief description of each dataset). The overall sample is formed by monthly data from 07/1964 to 06/2014 (600 months).

Table 1: List of datasets

Denomination	Composition criteria	Number of Securities
SBM6	Porfolios formed on size and book-to-market	6
FF10	Industry portfolios	10
SOP25	Portfolios formed on size and operating profitability	25
BMOP25	Portfolios formed on book-to-market and operating profitability	25

This table lists the acronym, composition criteria and number of portfolios of the four datasets used in the empirical work. For more information on these datasets consult the Fama/French benchmark collection in `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

The Pareto frontiers (in-sample) for the overall available period are reported in Subsection 6.1. The out-of-sample analysis of the skewness/semivariance efficient portfolios is described in Subsection 6.2.

6.1 In-sample analysis

We applied direct multisearch (see Appendix B) to determine the Pareto frontier of the skewness/semivariance biobjective optimization problem (9) for three different values of B: B_{ms}, the return of the maximum Sharpe ratio portfolio[1] (solution of problem (3)); B_{mv}, the return of the

[1] We considered as a risk-free asset the 90-day Treasury-Bills US. Such data is public and made available by the Federal Reserve, at the site `www.federalreserve.gov`.

minimum variance portfolio (solution of problem (5)); and $B_{1/N}$, the return of the $1/N$ porfolio (given by equation (6)).

Let $P_{t,i}$ be the price at time t of security i. The discrete return rate, at time t of security i, is given by

$$R_{t,i} = \frac{P_{t,i} - P_{t-1,i}}{P_{t-1,i}} = \frac{P_{t,i}}{P_{t-1,i}} - 1. \tag{10}$$

While the corresponding continuous return rate is given by

$$r_{t,i} = \ln\left(\frac{P_{t,i}}{P_{t-1,i}}\right) \tag{11}$$

Thus, the equivalence between these rates imply that

$$r_{t,i} = \ln\left(1 + R_{t,i}\right) \tag{12}$$

The monthly returns available in the Fama and French benchmark collection are discrete returns. Since discrete returns allow for an accurate cross-sectional aggregation while continuous returns allow for an accurate time aggregation, the initial returns are converted into continuous returns using equation (12) in order to obtain better parameter estimates.

Figures 1 to 3, Figures 4 to 6, Figures 7 to 9 and Figures 10 to 12 contain the plots of the Pareto frontiers, computed using the overall sample period, for the SBM6, FF10, SOP25 and BMOP25 datasets, respectively. Figures 1, 4, 7 and 10 correspond to the cases where the benchmark return is B_{ms}. Figures 2, 5, 8 and 11 correspond to the cases where the benchmark return is B_{mv}. Figures 3, 6, 9 and 12 correspond to the cases where the benchmark return is $B_{1/N}$.

As it is clear from the plots visualization, direct multisearch was able to determine the Pareto frontiers for the biobjective skewness/semivariance optimization problems (solutions of problem (9)) for all the instances considered. Thus, this methodology offers a direct way for analysing the efficient trade-off between skewness and semivariance.

6.2 Out-of-sample analysis

The validation of a new methodology for portfolio selection must undoubtedly be inferred from out-of-sample performance analysis. This subsection deals with an extensive out-of-sample analysis of the efficient skewness/semivariance portfolios, constructed according the model proposed in Section 5, and compared with each of the benchmark portfolios presented in Section 3.

The out-of-sample analysis relies on a rolling-sample approach. We considered an estimation window of 120 months. For an estimation window of 120 months, beginning from 07/1964 to 06/1974, there are 480 evaluation periods (months) until 06/2014. Firstly, the benchmark portfolios (the ms portfolio, the mv portfolio and the $1/N$ portfolio) are computed. Secondly, the Pareto frontiers of the skewness/semivariance biobjective optimization problem (9) are computed, considering B_{ms}, B_{mv} and $B_{1/N}$ as benchmark returns for the semivariance computation. Thirdly, three efficient skewness/semivariance portfolios in each of the in-sample Pareto frontiers are selected. The first one, w_{SSR}, is the portfolio that maximizes a skewness per semivariance ratio (SSR)

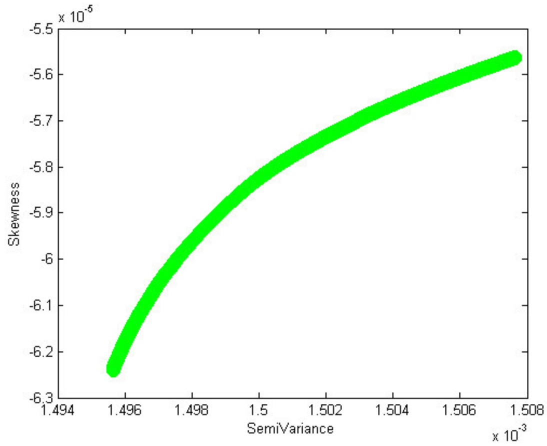

Figure 1: SBM6 skewness/semivariance efficient frontier, with B_{ms} as the benchmark return for semivariance computation.

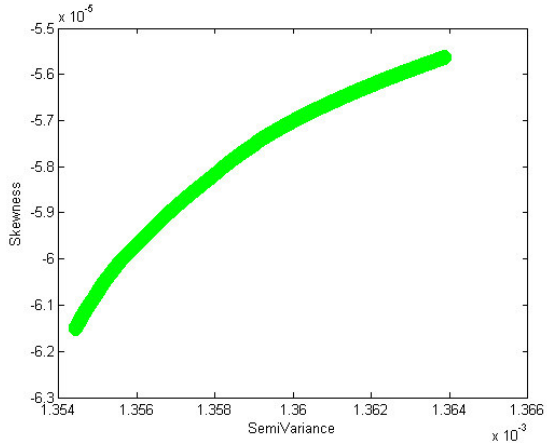

Figure 2: SBM6 skewness/semivariance efficient frontier, with B_{mv} as the benchmark return for semivariance computation.

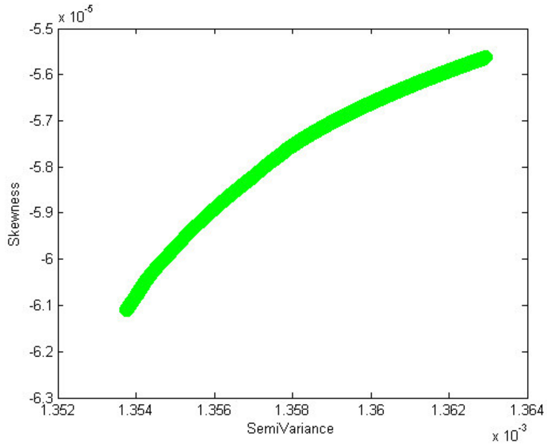

Figure 3: SBM6 skewness/semivariance efficient frontier, with $B_{1/N}$ as the benchmark return for semivariance computation.

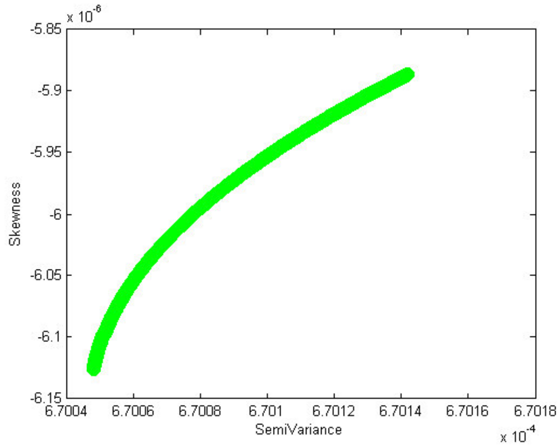

Figure 4: FF10 skewness/semivariance efficient frontier, with B_{ms} as the benchmark return for semivariance computation.

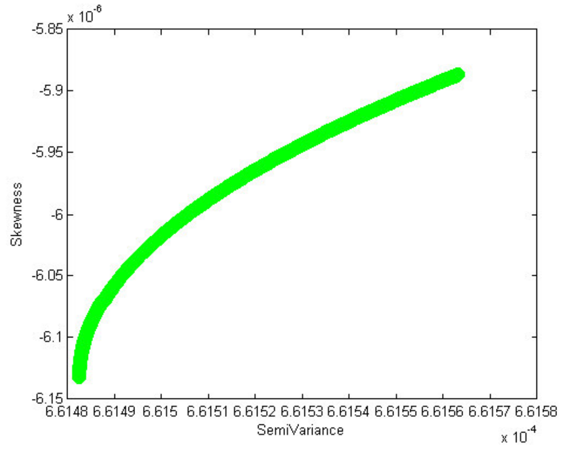

Figure 5: FF10 skewness/semivariance efficient frontier, with B_{mv} as the benchmark return for semivariance computation.

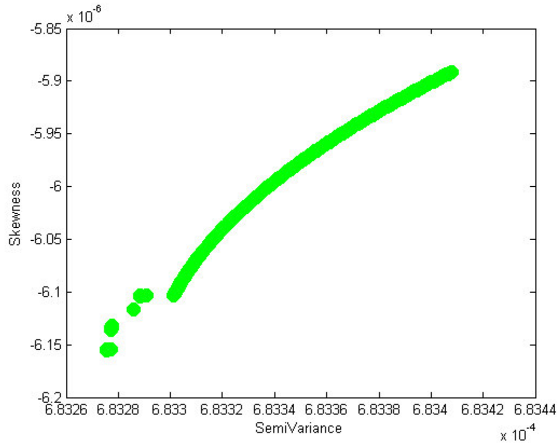

Figure 6: FF10 skewness/semivariance efficient frontier, with $B_{1/N}$ as the benchmark return for semivariance computation.

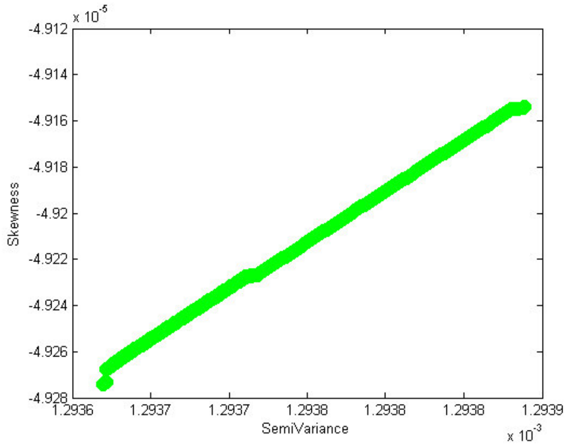

Figure 7: SOP25 skewness/semivariance efficient frontier, with B_{ms} as the benchmark return for semivariance computation.

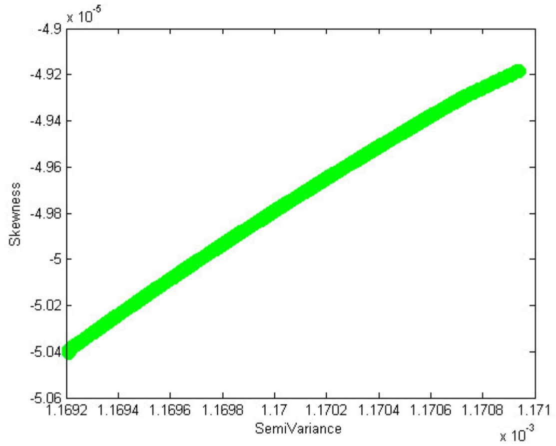

Figure 8: SOP25 skewness/semivariance efficient frontier, with B_{mv} as the benchmark return for semivariance computation.

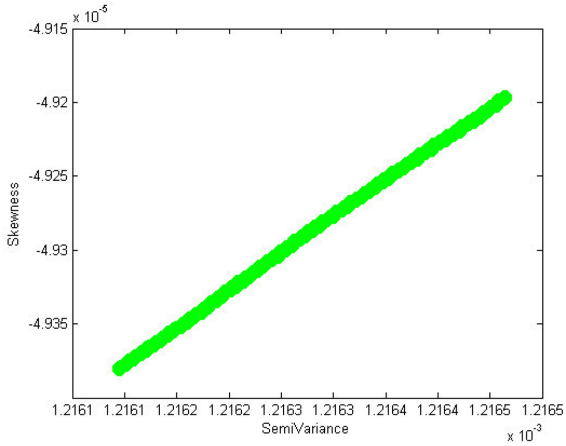

Figure 9: SOP25 skewness/semivariance efficient frontier, with $B_{1/N}$ as the benchmark return for semivariance computation.

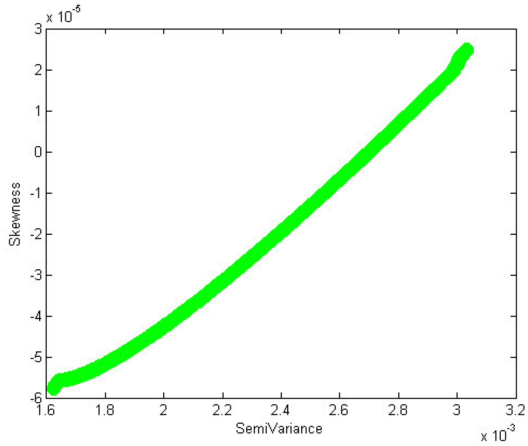

Figure 10: BMOP25 skewness/semivariance efficient frontier, with B_{ms} as the benchmark return for semivariance computation.

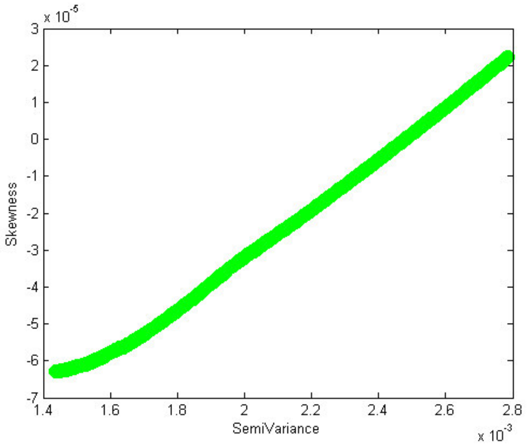

Figure 11: BMOP25 skewness/semivariance efficient frontier, with B_{mv} as the benchmark return for semivariance computation.

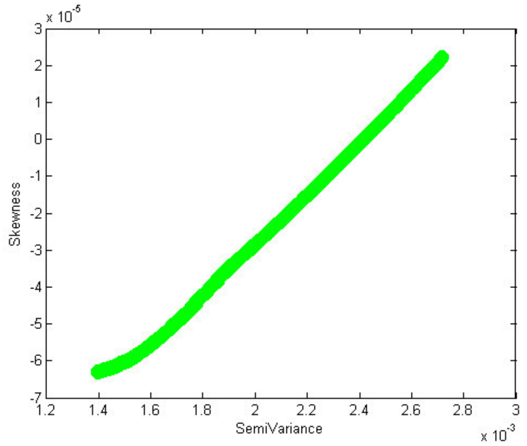

Figure 12: BMOP25 skewness/semivariance efficient frontier, with $B_{1/N}$ as the benchmark return for semivariance computation.

$$SSR = \frac{\kappa(w_{SSR})}{\Sigma_B(w_{SSR})}. \tag{13}$$

The second one, w_{SR}, is the portfolio that maximizes the Sharpe ratio (SR)

$$SR = \frac{\mu(w_{SR}) - r_f}{\sqrt{\nu(w_{SR})}}. \tag{14}$$

The third one, w_{SOR}, is the portfolio that maximizes the Sortino ratio (SOR)

$$SOR = \frac{\mu(w_{SOR}) - B}{\sqrt{\Sigma_B(w_{SOR})}}. \tag{15}$$

The numerator of these ratios for the chosen portfolios may be negative. Thus the denominators are modified as proposed by Israelsen (2005), in order to have correct rankings. Finally, each portfolio is held fixed and its returns (discrete returns) are observed over the next month. The process is repeated forward until the end of the overall sample, where at each time the estimation window is obtain from discarding the first month and joining the next month from the previous window. Since continuous returns allow for an accurate time aggregation, the monthly out-of-sample discrete returns are converted into continuous returns (according to equation (12)).

The out-of-sample performance is measured using four metrics: a skewness per semivariance ratio (Subsection 6.2.1), the Sharpe ratio (Subsection 6.2.2), the Sortino ratio (Subsection 6.2.3) and the turnover (Subsection 6.2.4).

6.2.1 Performance measured by a skewness per semivariance ratio

We computed an out-of-sample skewness per semivariance ratio, defined as the sample skewness, $\hat{\kappa}_B$, divided by their sample semivariance, $\hat{\Sigma}_B$:

$$\widehat{\mathrm{SSR}} = \frac{\widehat{\kappa}_B}{\widehat{\Sigma}_B}. \tag{16}$$

Then, we computed the bootstrap p-values of the difference between the skewness per semivariance ratio of each efficient skewness/semivariance portfolio from those of the benchmarks. Since none of the differences were statistically significant, we decided do not report these results here. Once the computed skewness per semivariance ratios are negative (we are in the presence of negative skewness), in order to achieve a correct rank of the portfolios, it is necessary to refine the ratios. We modified the denominator according the methodology proposed by Israelsen (2005). Thus we computed the refined skewness per semivariance ratio as

$$\widehat{\mathrm{SSR}}_{\mathrm{ref}} = \frac{\widehat{\kappa}_B}{\widehat{\Sigma}_B^{\widehat{\kappa}_B / abs(\widehat{\kappa}_B)}}, \tag{17}$$

where $abs(.)$ is the absolute value function.

Table 2 reports the refined skewness per semivariance ratios, when we choose as a benchmark return the maximum Sharpe ratio portfolio return (B_{ms}). We can see that the efficient skewness/semivariance portfolios w_{SR} and w_{SOR} have a higher refined skewness per semivariance ratio than two (the ms and the $1/N$ portfolios) of the three benchmarks portfolios, for all the datasets. For the datasets with the highest number of securities (SOP25 and BMOP25) all the

Table 2: Portfolio refined skewness per semivariance ratios for the benchmark return B_{ms}

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	-1.7917E-07	4	-6.3460E-08	4	-1.9356E-07	5	-2.0470E-07	5
mv portfolio	-1.3453E-07	1	-8.8977E-09	1	-7.6822E-08	1	-1.3521E-07	1
$1/N$ portfolio	-2.4616E-07	6	-2.7290E-07	6	-2.7886E-07	6	-2.9854E-07	6
Efficient skewness/semivariance portfolios								
w_{SSR}	-2.1696E-07	5	-9.0904E-08	5	-1.5191E-07	4	-1.8266E-07	4
w_{SR}	-1.5179E-07	2	-2.8133E-08	2	-1.1641E-07	2	-1.6610E-07	2
w_{SOR}	-1.7167E-07	3	-5.7460E-08	3	-1.2809E-07	3	-1.7596E-07	3

This table reports, for each of the datasets listed in Table 1, the monthly refined skewness per semivariance ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The benchmark return needed in the computation of the semivariance is the maximum Sharpe ratio portfolio return (B_{ms}). This table also reports the monthly refined skewness per semivariance ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). According with the refined skewness per semivariance ratios it is reported the correct rank of each portfolio.

Table 3: Portfolio refined skewness per semivariance ratios for the benchmark return B_{mv}

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	-1.5289E-07	4	-6.2869E-08	4	-1.7462E-07	5	-1.8242E-07	4
mv portfolio	-1.1256E-07	1	-8.7818E-09	1	-6.8100E-08	1	-1.1841E-07	1
$1/N$ portfolio	-2.0989E-07	6	-2.7083E-07	6	-2.5299E-07	6	-2.6599E-07	6
Efficient skewness/semivariance portfolios								
w_{SSR}	-1.8553E-07	5	-9.2381E-08	5	-1.4175E-07	4	-1.8467E-07	5
w_{SR}	-1.2960E-07	2	-2.8500E-08	2	-1.0657E-07	2	-1.5624E-07	2
w_{SOR}	-1.5066E-07	3	-5.1675E-08	3	-1.1587E-07	3	-1.6621E-07	3

This table reports, for each of the datasets listed in Table 1, the monthly refined skewness per semivariance ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The benchmark return needed in the computation of the semivariance is the minimum variance portfolio return (B_{mv}). This table also reports the monthly refined skewness per semivariance ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). According with the refined skewness per semivariance ratios it is reported the correct rank of each portfolio.

Table 4: Portfolio refined skewness per semivariance ratios for the benchmark return $B_{1/N}$

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	-1.6162E-07	4	-6.5469E-08	4	-1.8097E-07	5	-1.8555E-07	4
mv portfolio	-1.1985E-07	1	-9.2942E-09	1	-7.1022E-08	1	-1.2056E-07	1
$1/N$ portfolio	-2.2200E-07	6	-2.7991E-07	6	-2.6171E-07	6	-2.7021E-07	6
Efficient skewness/semivariance portfolios								
w_{SSR}	-1.9538E-07	5	-9.6091E-08	5	-1.4698E-07	4	-2.0206E-07	5
w_{SR}	-1.3590E-07	2	-2.9696E-08	2	-1.0937E-07	2	-1.5525E-07	2
w_{SOR}	-1.5376E-07	3	-6.0198E-08	3	-1.2314E-07	3	-1.7617E-07	3

This table reports, for each of the datasets listed in Table 1, the monthly refined skewness per semivariance ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The benchmark return needed in the computation of the semivariance is the $1/N$ portfolio return (B_{mv}). This table also reports the monthly refined skewness per semivariance ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). According with the refined skewness per semivariance ratios it is reported the correct rank of each portfolio.

efficient skewness/semivariance portfolios (w_{SSR}, w_{SR} and w_{SOR}) have a higher refined skewness per semivariance ratio than two (the *ms* and the $1/N$ portfolios) of the three benchmarks portfolios. The same pattern is found when we choose as a benchmark return the minimum variance portfolio return (B_{mv}) (see Table 3) and the $1/N$ portfolio return ($B_{1/N}$) (see Table 4).

These results suggests the robustness of the efficiency provided by the skewness/semivariance model.

6.2.2 Performance measured by the Sharpe ratio

Given the time series of monthly out-of-sample returns, for each portfolio we computed the out-of-sample Sharpe ratio, defined as the sample mean of excess returns (over the risk-free asset), $\hat{m}$, divided by its sample standard deviation, $\hat{\sigma}$:

$$\widehat{\mathrm{SR}} = \frac{\widehat{m}}{\widehat{\sigma}}. \tag{18}$$

The results are presented in Table 5. This table also reports the bootstrap p-values (Ledoit and Wolf, 2008) for the statistical significance of the difference between the Sharpe ratios of the benchmarks and the efficient skewness/semivariance portfolios.

For the SBM6 dataset, independent of the benchmark return used in the computation of the semivariance (B_{ms}, B_{mv} or $B_{1/N}$), the efficient skewness/semivariance portfolio w_{SOR} has a higher Sharpe ratio than all the three benchmark portfolios (the *ms* portfolio, the *mv* portfolio and the $1/N$ portfolio). For all the benchmark returns (B_{ms}, B_{mv} and $B_{1/N}$), the difference between the Sharpe ratios of the efficient skewness/semivariance portfolio w_{SOR} and the $1/N$ portfolio is statistically significant.

Table 5: Out-of-sample Sharpe ratios

Benchmark	Strategy	SBM6	FF10	SOP25	BMOP25
	ms portfolio	0.2572	0.2736	0.2113	0.2993
	mv portfolio	0.2480	0.3188	0.2130	0.2935
	$1/N$ portfolio	0.2235	0.2115	0.1976	0.2357
		Efficient skewness/semivariance portfolios			
B_{ms}	w_{SSR}	0.2541	0.2689	0.2041	0.2624
		$(0.89)^1(0.74)^2(0.15)^3$	$(0.92)^1(0.11)^2(0.16)^3$	$(0.75)^1(0.63)^2(0.74)^3$	$(0.16)^1(0.25)^2(0.26)^3$
	w_{SR}	0.2472	0.3078	0.2071	0.3151
		$(0.52)^1(0.95)^2(0.19)^3$	$(0.35)^1(0.59)^2(0.02)^3$	$(0.84)^1(0.70)^2(0.61)^3$	$(0.23)^1(0.14)^2(0.00)^3$
	w_{SOR}	0.2640	0.2829	0.2125	0.3019
		$(0.73)^1(0.38)^2(0.03)^3$	$(0.81)^1(0.21)^2(0.07)^3$	$(0.95)^1(0.98)^2(0.44)^3$	$(0.89)^1(0.69)^2(0.00)^3$
B_{mv}	w_{SSR}	0.2539	0.2676	0.2018	0.2596
		$(0.88)^1(0.74)^2(0.14)^3$	$(0.89)^1(0.11)^2(0.17)^3$	$(0.68)^1(0.54)^2(0.84)^2$	$(0.13)^1(0.20)^2(0.31)^3$
	w_{SR}	0.2485	0.3057	0.2077	0.3109
		$(0.58)^1(0.96)^2(0.17)^3$	$(0.38)^1(0.51)^2(0.02)^3$	$(0.86)^1(0.74)^2(0.57)^3$	$(0.37)^1(0.22)^2(0.00)^3$
	w_{SOR}	0.2629	0.2842	0.2097	0.3051
		$(0.77)^1(0.38)^2(0.03)^3$	$(0.80)^1(0.22)^2(0.06)^3$	$(0.94)^1(0.86)^2(0.51)^3$	$(0.71)^1(0.55)^2(0.00)^3$
$B_{1/N}$	w_{SSR}	0.2540	0.2658	0.2004	0.2566
		$(0.89)^1(0.75)^2(0.15)^3$	$(0.87)^1(0.11)^2(0.19)^3$	$(0.63)^1(0.51)^2(0.89)^3$	$(0.10)^1(0.17)^2(0.37)^3$
	w_{SR}	0.2488	0.3050	0.2057	0.3127
		$(0.61)^1(0.94)^2(0.17)^3$	$(0.39)^1(0.48)^2(0.02)^3$	$(0.78)^1(0.66)^2(0.65)^3$	$(0.28)^1(0.19)^2(0.00)^3$
	w_{SOR}	0.2634	0.2806	0.2099	0.3073
		$(0.76)^1(0.39)^2(0.03)^3$	$(0.87)^1(0.19)^2(0.09)^3$	$(0.95)^1(0.86)^2(0.51)^3$	$(0.61)^1(0.50)^2(0.00)^3$

This table reports, for each of the datasets listed in Table 1, the monthly Sharpe ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. This table also reports the monthly Sharpe ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). In the computation of the semivariance it is considered three different values for the benchmark return: the maximum Sharpe ratio portfolio return (B_{ms}), the minimum variance portfolio return (B_{mv}) and the $1/N$ portfolio return ($B_{1/N}$). In parenthesis are the bootstrap p-values of the difference between the Sharpe ratio of each efficient skewness/semivariance portfolio from those of the benchmarks: $(.)^1$ is from the ms portfolio, $(.)^2$ is from the mv portfolio and $(.)^3$ is from the $1/N$ portfolio; these p-values are computed according the Ledoit and Wolf (Ledoit and Wolf, 2008) methodology.

In the case of the FF10 dataset, for all the benchmark returns (B_{ms}, B_{mv} and $B_{1/N}$), the efficient skewness/semivariance portfolios w_{SR} and w_{SOR} have a higher Sharpe ratio than two (the *ms* portfolio and the $1/N$ portfolio) of the three benchmark portfolios. The difference between the Sharpe ratio of the efficient skewness/semivariance portfolio w_{SR} and the benchmark $1/N$ portfolio is always statistically significant.

We don't observe statistically significant differences, between the Sharpe ratios of the efficient skewness/semivariance portfolios and the benchmark portfolios, for the SOP25 dataset. However, we can see that in the case that we compute the semivariance using as a benchmark return B_{ms}, the efficient skewness/semivariance portfolio w_{SOR} has a higher Sharpe ratio than two (the ms portfolio and the $1/N$ portfolio) of the three benchmark portfolios.

Finally, for the BMOP25 dataset, independent of the benchmark return used in the computation of the semivariance (B_{ms}, B_{mv} or $B_{1/N}$), the efficient skewness/semivariance portfolios w_{SR} and w_{SOR} have a higher Sharpe ratio than all the benchmark portfolios. The differences between the Sharpe ratios of these two efficient skewness/semivariance portfolios (w_{SR} and w_{SOR}) and the benchmark $1/N$ portfolio are always statistically significant.

These results show that the efficient skewness/semivariance portfolios are consistently competitive, and often superior, comparatively to the benchmark portfolios. The efficient skewness/semivariance portfolios consistently outperform the $1/N$ benchmark portfolio. To achieve a higher Sharpe ratio, this analysis suggests that one should choose a benchmark return (for the computation of the semivariance) according the specific nature of the data.

6.2.3 Performance measured by the Sortino ratio

We computed the out-of-sample Sortino ratio, defined as the sample mean of out-of-sample excess returns (over the benchmark return B), $\hat{m}_B$, divided by their sample standard semideviation, $\hat{\sigma}_B$:

$$\widehat{\mathrm{SOR}} = \frac{\widehat{m}_B}{\widehat{\sigma}_B}. \tag{19}$$

Then, we computed the bootstrap p-values of the difference between the Sortino ratio of each efficient skewness/semivariance portfolio from those of the benchmarks. Since none of the differences were statistically significant, we decided do not report these results here. For the data analysed we are in the presence of negative excess returns. Thus, in order to achieve a correct rank of the portfolios considered, we modified the denominator according with the methodology proposed by Israelsen (2005). We then computed the refined Sortino ratio as

$$\widehat{\mathrm{SOR}}_{\mathrm{ref}} = \frac{\widehat{m}_B}{\widehat{\sigma}_B^{\widehat{m}_B/abs(\widehat{m}_B)}}, \tag{20}$$

where $abs(.)$ is the absolute value function.

Table 6 reports the refined Sortino ratios when we choose as a benchmark return for the computation of the semivariance, the maximum Sharpe ratio portfolio return (B_{ms}). We can see that for the SBM6 dataset, the efficient skewness/semivariance portfolios w_{SSR} and w_{SOR} have a higher refined Sortino ratio than all the benchmark portfolios. In the case of the FF10 dataset, the efficient skewness/semivariance portfolios have a higher refined Sortino ratio than one (the mv portfolio) of the benchmark portfolios. The efficient skewness/semivariance portfolio w_{SOR}, in the SOP25 dataset, has a higher refined Sortino ratio than two (the mv portfolio and the $1/N$ portfolio) of the three benchmark portfolios. For the BMOP25 dataset, the efficient skewness/semivariance portfolio w_{SR} has the highest refined Sortino ratio among all the portfolios.

In Table 7 we can find the refined Sortino ratios for the case in which the benchmark return for the computation of the semivariance is the minimum variance portfolio return (B_{mv}). For the SBM6 dataset, the efficient skewness/semivariance portfolio w_{SOR} has the highest refined Sortino ratio among all the portfolios. In the cases of the FF10 and SOP25 datasets, the efficient skewness/semivariance portfolios have a higher refined Sortino ratio than one (the mv portfolio)

Table 6: Portfolio refined Sortino ratios for the benchmark return B_{ms}

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	-1.8509E-04	3	0.0026	1	-1.7974E-04	1	-1.3856E-04	2
mv portfolio	-2.3453E-04	5	-3.8627E-05	6	-1.9855E-04	6	-2.1214E-04	5
$1/N$ portfolio	-2.5861E-04	6	-2.0271E-05	2	-1.8833E-04	3	-2.7621E-04	6
Efficient skewness/semivariance portfolios								
w_{SSR}	-1.6812E-04	1	-2.6925E-05	4	-1.9163E-04	4	-1.7927E-04	4
w_{SR}	-2.2766E-04	4	-2.0322E-05	3	-1.9606E-04	5	-1.2620E-04	1
w_{SOR}	-1.7483E-04	2	-2.8158E-05	5	-1.8310E-04	2	-1.4281E-04	3

This table reports, for each of the datasets listed in Table 1, the monthly refined Sortino ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The benchmark return needed in the computation of the semivariance is the maximum Sharpe ratio portfolio return (B_{ms}). This table also reports the monthly refined Sortino ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). According with the refined Sortino ratios it is reported the correct rank of each portfolio.

Table 7: Portfolio refined Sortino ratios for the benchmark return B_{mv}

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	0.0625	2	0.0128	1	-4.8419E-06	2	0.0480	2
mv portfolio	0.0257	6	-2.9303E-05	6	-3.7515E-05	6	-1.0974E-05	5
$1/N$ portfolio	0.0237	5	-5.2105E-06	2	-2.1300E-07	1	-3.6604E-05	6
Efficient skewness/semivariance portfolios								
w_{SSR}	0.0608	3	-2.0269E-05	5	-2.0292E-05	4	0.0266	4
w_{SR}	0.0355	4	-1.3347E-05	3	-2.5222E-05	5	0.0495	1
w_{SOR}	0.0665	1	-1.8270E-05	4	-1.8133E-05	3	0.0479	3

This table reports, for each of the datasets listed in Table 1, the monthly refined Sortino ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The benchmark return needed in the computation of the semivariance is the minimum variance portfolio return (B_{mv}). This table also reports the monthly refined Sortino ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). According with the refined Sortino ratios it is reported the correct rank of each portfolio.

of the benchmark portfolios. For the BMOP25 dataset, the efficient skewness/semivariance portfolio w_{SR} has the highest refined Sortino ratio among all the portfolios.

Table 8: Portfolio refined Sortino ratios for the benchmark return $B_{1/N}$

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	4.6247E-04	2	-3.6611E-05	1	-6.3687E-05	2	0.0297	3
mv portfolio	-5.5112E-05	5	-7.0521E-05	3	-9.1409E-05	6	-3.6877E-05	5
$1/N$ portfolio	-6.0225E-05	6	-7.1192E-05	4	-6.3626E-05	1	-6.7692E-05	6
Efficient skewness/semivariance portfolios								
w_{SSR}	1.6298E-04	3	-7.3915E-05	5	-8.2534E-05	4	0.0062	4
w_{SR}	-4.0554E-05	4	-5.8940E-05	2	-8.5081E-05	5	0.0335	2
w_{SOR}	0.0052	1	-7.5954E-05	6	-7.3910E-05	3	0.0339	1

This table reports, for each of the datasets listed in Table 1, the monthly refined Sortino ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The benchmark return needed in the computation of the semivariance is the $1/N$ portfolio return ($B_{1/N}$). This table also reports the monthly refined Sortino ratios for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). According with the refined Sortino ratios it is reported the correct rank of each portfolio.

Finally, in Table 8 we find the refined Sortino ratios for the case in which the benchmark return is the $1/N$ portfolio return ($B_{1/N}$). The efficient skewness/semivariance portfolio w_{SOR} have the highest refined Sortino ratio, among all the portfolios, in two cases (for the SBM6 dataset and for the BMOP25 dataset). For the FF10 dataset, the efficient skewness/semivariance portfolio w_{SR} has a higher refined Sortino ratio than two (the *mv* and the $1/N$ portfolios) of the three benchmark portfolios. In the case of the SOP25 dataset, the efficient skewness/semivariance portfolios have a higher refined Sortino ratio than one (the *mv* portfolio) of the benchmark portfolios.

Once again, these results suggests the robustness of the efficiency provided by the skewness/semivariance model.

6.2.4 Performance measured by the turnover

Given the time series of monthly out-of-sample returns, for each portfolio considered, we computed the portfolio turnover, defined as the average sum of the absolute value of the trades across the N available securities:

$$\text{turnover} = \frac{1}{\#periods} \sum_{t=1}^{\#periods} \sum_{i=1}^{N} \left(\left| w_{t+1,i} - w_{t,i}^{h} \right| \right), \tag{21}$$

where $w_{t+1,i}$ is the portfolio weight after rebalancing at time $t+1$ and $w_{t,i}^h$ is the portfolio weight before rebalancing at time $t+1$. Thus, $w_{t,i}^h$, is computed as

$$w_{t,i}^h = w_{t-1,i} \frac{1 + r_{t,i}}{1 + r_{t,p}},$$

where $r_{t,i}$ is the return at time t of the security i and $r_{t,p}$ is the return at time t of the portfolio. The results are reported in Table 9.

Table 9: Portfolio turnovers

Benchmark	Strategy	SBM6	FF10	SOP25	BMOP25
	ms portfolio	0.0779	0.0660	0.2483	0.1656
	mv portfolio	0.0318	0.0167	0.0630	0.0549
	$1/N$ portfolio	0.0151	0.0217	0.0156	0.0171
Efficient skewness/semivariance portfolios					
	w_{SSR}	0.0854	0.1183	0.2566	0.1936
B_{ms}	w_{SR}	0.1162	0.0981	0.2456	0.2038
	w_{SOR}	0.0830	0.1174	0.2460	0.2091
	w_{SSR}	0.0805	0.1134	0.2471	0.1756
B_{mv}	w_{SR}	0.1034	0.0875	0.2549	0.1996
	w_{SOR}	0.0926	0.1057	0.2566	0.2047
	w_{SSR}	0.0800	0.1117	0.2511	0.1805
$B_{1/N}$	w_{SR}	0.1096	0.0930	0.2633	0.2050
	w_{SOR}	0.0921	0.1110	0.2540	0.2112

This table reports, for each dataset listed in Table 1, the monthly turnovers for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. This table also reports the monthly turnovers for the efficient skewness/semivariance portfolios referred in Section 6.2: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). In the computation of the semivariance it is considered three different values for the benchmark return: the maximum Sharpe ratio portfolio return (B_{ms}), the minimum variance portfolio return (B_{mv}) and the $1/N$ portfolio return ($B_{1/N}$).

From Table 9 we can observe that for all datasets, except for the FF10 dataset where the minimum variance portfolio has the lower turnover, the turnover of the $1/N$ portfolio is lower than all the others portfolios analysed.

For the SBM6 dataset the efficient skewness/semivariance portfolio that has the lower turnover is the maximum skewness per semivariance portfolio (w_{SSR}) constructed using as a benchmark return the $1/N$ portfolio return ($B_{1/N}$).

In turn, for the FF10 dataset the efficient skewness/semivariance portfolio that has the lower turnover is the maximum Sharpe ratio portfolio (w_{SR}) constructed using as a benchmark return the minimum variance portfolio return (B_{mv}).

For the SOP25 dataset the efficient skewness/semivariance portfolio that has the lower turnover is the maximum Sharpe ratio portfolio (w_{SR}) constructed using as a benchmark return the maximum Sharpe ratio portfolio return (B_{ms}).

Finally, for the BMOP25 dataset the efficient skewness/semivariance portfolio that has the lower turnover is the maximum skewness per semivariance portfolio (w_{SSR}) constructed using as a benchmark return the minimum variance portfolio return (B_{mv}).

Note that all the lower turnover efficient skewness/semivariance portfolios, mentioned in the last five paragraphs, exhibit a turnover very similar to the turnover of one (the ms portfolio) of the three benchmark portfolios. In general, we can note that the efficient skewness/semivariance portfolios constructed using as a benchmark return (B_{ms}) are the portfolios with the highest turnovers.

7 Conclusions and future work

We have proposed a new model to measure and deal with risk in portfolio selection. We propose a direct analysis of the efficient trade-off between skewness and semivariance through a skewness/semivariance biobjective optimization problem. We computed skewness as a third moment tensor. We overcame the endogeneity problem of the cosemivariance matrix using, according to our knowledge, for the first time, a derivative-free algorithm. The solver chosen for solving the skewness/semivariance biobjective optimization problem is based on direct multisearch. Direct-search methods based on polling are known extremely robust due their directional properties. We have observed the robustness of direct multisearch in the four empirical datasets collected from the Fama/French benchmark collection, since direct multisearch was capable of determining in-sample the Pareto frontier for the biobjective skewness/semivariance problem, taking into account for the computation of the semivariance, three different benchmarks returns.

In addiction, we performed an extensive out-of-sample analysis. The results showed that, although the efficient skewness/semivariance portfolios exhibited a higher turnover compared to the benchmark portfolios, they are consistently competitive and often superior, in terms of out-of-sample Sharpe ratio. A surprising fact was that the efficient skewness/semivariance portfolios, consistently outperforms the $1/N$ portfolio in terms of out-of-sample Sharpe ratio. The efficient skewness/semivariance portfolios, also exhibited a consistently competitive good performance in terms of skewness per semivariance ratio and Sortino ratio, which suggests the robustness of the efficiency provided by the skewness/semivariance model.

In the analysis performed, we have just used three different benchmark returns for the computation of the semivariance. Other choices could have been made (e.g. the risk-free return, an index return). It is clear that the choice of this parameter, depending on the nature of the data, has a profound impact in the results. Within the skewness/semivariance model, the investor has the freedom to choose this parameter according to any pre-established criteria. Besides, we only evaluate the performance of three efficient skewness/semivariance portfolios (chosen according to three different criteria). Other efficient portfolios could have been chosen according to the investor's preferences.

In the proposed model, we did not consider other constraints than non short-selling that can improve portfolio performance. One can introduce, for example, turnover constraints in order to control the transaction costs. Another promising possibility is to introduce constraints to explore information about the cross-sectional characteristics of securities, as presented in Brandt et al. (2009). The proposed skewness/semivariance model is ready to incorporate such a constraints.

A Proof for Proposition 3.1

Proof. The proof is based on a lifting technique. Looking to the objective function of problem (3)

$$f(w) = \frac{m^\top w - r_f}{\sqrt{w^\top M_2 w}},$$

the numerator, $m^\top w - r_f$, can be rewritten as $(m - r_f e)^\top w$, since $e^\top w = 1$. Consider now the change of variable on the form

$$x = \tau w \quad \text{and} \quad (m - r_f e)^\top w = \frac{1}{\tau}, \quad \text{for some } \tau > 0.$$

Thus, the problem of maximizing the Sharpe ratio (problem (3)) is equivalent to the following problem on the variables (x, τ)

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^N, \tau \in \mathbb{R}^+} & \dfrac{1}{\sqrt{x^\top M_2 x}} \\ \text{subject to} & (m - r_f e)^\top x = 1, \\ & e^\top x = \tau, \\ & \tau > 0. \\ & x_i \geq 0, \quad i = 1, \ldots, N. \end{array}$$

Since maximize $\dfrac{1}{\sqrt{x^\top M_2 x}}$ it is equivalent to minimize $x^\top M_2 x$, we have that problem (3) it is equivalent to the following convex QP

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^N, \tau \in \mathbb{R}^+} & x^\top M_2 x \\ \text{subject to} & (m - r_f e)^\top x = 1, \\ & e^\top x = \tau, \\ & \tau > 0. \\ & x_i \geq 0, \quad i = 1, \ldots, N. \end{array}$$

■

B Direct multisearch for multiobjective optimization

B.1 Multiobjective optimization

In a multiobjective optimization problem one optimizes 'simultaneously' multiple objective functions (sometimes 'conflicting'). A constrained nonlinear multiobjective problem can be written in the form:

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & F(x) \equiv (f_1(x), \ldots, f_m(x))^\top \\ \text{subject to} & x \in \Omega \subset \mathbb{R}^n, \end{array} \tag{22}$$

involving m objective functions or objective function components $f_i : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}, i = 1, \ldots, m$, and a feasible region Ω.

In the presence of several objective functions, the minimizers of one function are not necessarily the minimizers of another function. To define some sort of optimality, it is crucial to have a way of comparing different points, such as the concept of Pareto dominance. Given two points x, y in Ω, we say that $x \prec y$ (x dominates y) if and only if

$$F(x) \prec_F F(y) \Longleftrightarrow F(y) - F(x) \in \mathbb{R}^m_+ \setminus \{0\},$$

where $\mathbb{R}^m_+$ is the nonnegative orthant $\mathbb{R}^m_+ = \{z \in \mathbb{R}^m : z \geq 0\}$. Note that the use of the nonnegative orthant induces a strict partial order.

A set of points in Ω is nondominated when no point in the set is dominated by another one in the set. The concept of Pareto dominance is thus used to characterize optimality in multiobjective optimization, by defining the set of Pareto optimizers or nondominated points. More formally, a point $x_* \in \Omega$ is said to be a (global) Pareto optimizer or a nondominated point of F in Ω if there is no $y \in \Omega$ such that $y \prec x_*$. The Pareto frontier or efficient frontier is the mapping by F of such set of Pareto optimizers.

B.2 A description of direct multisearch

Direct multisearch is a class of methods for the solution of multiobjective optimization problems of the form (22) without the use of derivatives, which does not aggregate or scalarize any of the objective function components. It essentially generalizes all direct-search methods of directional type from single to multiobjective optimization. As in direct search (Conn et al., 2009, Kolda et al., 2003), each iteration of direct multisearch is organized around a search step and a poll step, where the latest is the one responsible for the convergence properties. The search step is optional and when included it aims to improving numerical performance. Direct multisearch tries, however, to capture the whole Pareto frontier from the polling procedure itself.

These methods maintain a list of feasible nondominated points. In both search and poll steps, the objective function components are evaluated at a finite set of points. A new trial list is then formed after adding the new points to the current list and removing all dominated ones that might have appeared. Successful iterations correspond then to list changes, i.e., to iterations where the trial list differs from the current one.

Poll centers are chosen from the list. Note that each point in the list is associated with a step size parameter. Polling consists of evaluating the function at points obtained by adding to the poll center a multiple (defined by the step size) of a set of directions. These directions typically form a positive spanning set (in other words, a set of directions that spans $\mathbb{R}^n$ with nonnegative coefficients).

In direct multisearch, constraints are handled using an extreme barrier function

$$F_\Omega(x) = \begin{cases} F(x) & \text{if } x \in \Omega, \\ (+\infty, \ldots, +\infty)^\top & \text{otherwise,} \end{cases}$$

which prevents adding infeasible points to the current list of nondominated points.

Direct multisearch is described below following the algorithmic framework in Custódio et al. (2011). A number of details are omitted and the reader is referred to Custódio et al. (2011) for a complete description. In particular, we omitted the two known globalization strategies (generation of points in integer lattices and imposition of sufficient decrease), under which the algorithm is known to be globally convergent, in the sense of yielding some form of convergence independently of the starting point or starting list. In practice, these globalization strategies amount to very minor modifications to the algorithm.

Algorithm B.1 (Direct Multisearch for Multiobjective Optimization)

Initialization

Choose $x_0 \in \Omega$ with $f_i(x_0) < +\infty, \forall i \in \{1, \ldots, m\}$, $\alpha_0 > 0$, $0 < \beta_1 \leq \beta_2 < 1$, and $\gamma \geq 1$. Initialize the list of nondominated points and corresponding step size parameters ($L_0 = \{(x_0; \alpha_0)\}$ in case of a singleton).

For $k = 0, 1, 2, \ldots$

1. **Selection of an iterate point:** Order the list L_k in some way and select the first item $(x; \alpha) \in L_k$ as the current iterate and step size parameter (thus setting $(x_k; \alpha_k) = (x; \alpha)$).
2. **Search step:** Compute a finite set of points $\{z_s\}_{s \in S}$ and evaluate F_Ω at each element. Set $L_{add} = \{(z_s; \alpha_k), s \in S\}$.
   Form L_{trial} by eliminating dominated points from $L_k \cup L_{add}$. If $L_{trial} \neq L_k$ declare the iteration (and the search step) successful, set $L_{k+1} = L_{trial}$, and skip the poll step.
3. **Poll step:** Choose a positive spanning set D_k. Evaluate F_Ω at the set of poll points $P_k = \{x_k + \alpha_k d : d \in D_k\}$. Set $L_{add} = \{(x_k + \alpha_k d; \alpha_k), d \in D_k\}$.
   Form L_{trial} by eliminating dominated points from $L_k \cup L_{add}$. If $L_{trial} \neq L_k$ declare the iteration (and the poll step) as successful and set $L_{k+1} = L_{trial}$. Otherwise, declare the iteration (and the poll step) unsuccessful and set $L_{k+1} = L_k$.
4. **Step size parameter update:** If the iteration was successful then maintain or increase the corresponding step size parameters: $\alpha_{k,new} \in [\alpha_k, \gamma\alpha_k]$ and replace all the new points $(x_k + \alpha_k d; \alpha_k)$ in L_{k+1} by $(x_k + \alpha_k d; \alpha_{k,new})$, when success is coming from the poll step, or $(z_s; \alpha_k)$ in L_{k+1} by $(z_s; \alpha_{k,new})$, when success is coming from the search; replace also $(x_k; \alpha_k)$, if in L_{k+1}, by $(x_k; \alpha_{k,new})$.
   Otherwise decrease the step size parameter: $\alpha_{k,new} \in [\beta_1\alpha_k, \beta_2\alpha_k]$ and replace the poll pair $(x_k; \alpha_k)$ in L_{k+1} by $(x_k; \alpha_{k,new})$.

The goal of direct multisearch is to approximate the true Pareto frontier, although theoretically one is only able to prove that there is a limit point in a stationary form of this front, as no aggregation or scalarization technique is incorporated. In fact, it is possible to prove under standard assumptions that direct multisearch (globalized by integer lattices or sufficient decrease) generates a sequence of (unsuccessful) iterates driving the step size to zero and converging to a candidate for Pareto minimizer of the original problem, meaning to a point that satisfies some form of Pareto stationary. Essentially, one is able to prove at such a limit point x_* that, given a direction d (in the cone tangent to Ω at x_*), there exists at least one objective function component $j \in \{1, \ldots, m\}$ such that $f_j^\circ(x_*; d) \geq 0$. Here the directional derivative $f_j^\circ(x_*; d)$ is defined in the Clarke (1990) way if f_j is only assumed Lipschitz continuous near x_*. The set of directions used by the algorithm (contained in the positive spanning sets D_k) must then be asymptotically dense in the unit sphere for x_* to be Pareto-Clarke stationary (Custódio et al., 2011). Such a result can be further generalized for discontinuous objective functions following the steps in Vicente and Custódio (2012).

The comprehensive numerical experience reported in Custódio et al. (2011) showed that direct multisearch performed better than all other derivative-free solvers for multiobjective optimization, even when using a relatively simple implementation with an empty search step. In particular, direct multisearch clearly outperformed the very popular NSGA-II (Deb et al., 2002). This benchmarking was done in a test set of more than 100 problems, among them some with discontinuous and nonconvex Pareto frontiers. A number of tools and metrics were used to summarize the numerical findings, including data and performance profiles for the presentation of the results and purity and spread metrics to measure the quality of the obtained Pareto frontiers.

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