Lie symmetries for a multi-component NLS Equation in $2+1$ dimensions

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Invariance of a differential equation under a set of transformations is equivalent to the existence of symmetries. The study of symmetries represents a fundamental aspect related to the analysis of integrability of differential equations, since this invariance property may be used to achieve partial or complete integration of such equations [8].

The basis of the theory of Lie symmetries lies in the invariance of differential equations under one-parameter transformations of their variables, [7, 8]. These transformations form a local Lie group of transformations, which depend on a continuous parameter, and project any solution of the equation into another solution. Besides, a standard method to find solutions can be implemented by using Lie symmetries: each symmetry leads to a similarity reduction for the PDE which allows to reduce by one the number of independent variables.

Furthermore, as it is well known, a PDE is considered integrable when it can be derived through the Lax equation associated to a spectral problem. Lie symmetries for PDEs are very popular in literature, however, Lie symmetries for Lax pairs are much less frequent [4]. Nevertheless, the determination of the symmetries of the Lax pair has the benefit that the reduction associated to each symmetry of the Lax pair provides not only the reduction of the fields, but also the reductions of the eigenfunctions and the spectral parameter.

This talk is based on the work [1] done by authors. This paper is devoted to the study of an integrable $(2+1)$-dimensional multi-component nonlinear Schrödinger equation

$$\begin{aligned} &i\vec{\alpha}_t + \vec{\alpha}_{xx} + 2\, m_x\, \vec{\alpha} = 0, \quad -i\vec{\alpha}^{\dagger}_t + \vec{\alpha}^{\dagger}_{xx} + 2\, m_x\, \vec{\alpha}^{\dagger} = 0, \\ &\left(m_y + \vec{\alpha}\, \vec{\alpha}^{\dagger}\right)_x = 0, \end{aligned} \tag{1}$$

where $\vec{\alpha}(x,y,t) = (\alpha_1(x,y,t),\ \alpha_2(x,y,t))^{\intercal}$ and $\vec{\alpha}^{\dagger}$ is the complex conjugate of $\vec{\alpha}$. $m(x,y,t)$ is a real scalar function related to the probability density $\vec{\alpha}\,\vec{\alpha}^{\dagger}$.

The reduction $x = y$ of (1) yields the Manakov system, which is often also called vector NLS system [6]. Integrability properties of this Manakov system are described in [5, 9], and different generalizations of this system and their solutions have been studied in [2, 3].

The associated linear problem, [1], can be written by means of the following three-component Lax pair (and their complex conjugates), with $\vec{\Psi}(x,y,t) = (\psi(x,y,t),\ \chi(x,y,t),$

$\rho(x,y,t))^\intercal$,

$$\begin{aligned} \psi_y &= -\,\alpha_1^\dagger\,\chi - \,\alpha_2^\dagger\,\rho, & \psi_t &= -\psi_{xx} - 2\,m_x\,\psi \\ \chi_x &= -\,\alpha_1\,\psi, & \chi_t &= -\left(\alpha_1\right)_x\,\psi + \alpha_1\,\psi_x \\ \rho_x &= -\,\alpha_2\,\psi, & \rho_t &= -\left(\alpha_2\right)_x\,\psi + \alpha_2\,\psi_x \end{aligned} \tag{2}$$

The Lie point symmetries of the system (1) and its Lax pair (2) have been determined. The resulting symmetries include nine arbitrary functions of the independent variables and a single arbitrary constant, which plays the role of the spectral parameter when the spectral problem is reduced to $1+1$ dimensions.

The commutation relations among the generators associated to each symmetry have been widely analyzed. Although the set of symmetries does not form a Lie algebra, these relations are consistent and closed. Eventually, we could define the Lie algebra associated to particular selections for the arbitrary functions.

Three non-trivial reductions to $1+1$ dimensions have been identified. The reduced equations and the reduced spectral problem have been simultaneously obtained. It is important to notice that the spectral parameter arises naturally in the process of constructing the reductions, due to the symmetry procedure itself.

References

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