BOOKLET OF ABSTRACTS

13TH INTERNATIONAL YOUNG RESEARCHERS WORKSHOP ON GEOMETRY, MECHANICS AND CONTROL

Coimbra, 6-8 December 2018
Department of Mathematics
University of Coimbra

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- Geir Bogfjellmo, *Symplectic integrators on* $S_2^n \times T^*R^m$
- Eduardo Fernández-Saiz, *Poisson-Hopf algebras in Lie-Hamilton systems*
- Iván Gutiérrez-Sagredo, *Drinfel'd doubles and coisotropic spacetimes*
- Hildeberto Jardón-Kojakhmetov, *Stabilisation of slow-fast systems at non-hyperbolic points*
- Víctor M. Jiménez, *Groupoids and Distributions: A new way to deal with non-uniform material bodies*
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- Clément Moreau, *Necessary Condition for Local Controllability*
- Rodrigo Takuro Sato Martín de Almagro, *Numerical optimal control of nonholonomic systems*
- Marcin Zajac, *Geometrical framework of gauge theories*

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- Miguel García-Fernández, *The reveal of minimal ruled surfaces using control theory*
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MINI-COURSES

On the topology of bracket-generating distributions

From THU to SAT

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A smooth manifold M can be endowed with many different geometric structures, many of which interact in interesting ways with the topology of the manifold itself. This is the case, for instance, of metrics with prescribed curvature, tight contact structures, or taut foliations.

Distributions are a type of geometric structure that arise naturally in Physics and, more particularly, in Control Theory. They are simply a (constant rank and smoothly varying) choice of linear subspace $\xi_p \subset T_pM$ at each $p \in M$. We can imagine that a particle in M is only allowed to move along the directions determined by the distribution ξ; as such, they are suitable for modelling systems with (linear) constraints.

Distributions, much like metrics, possess many pointwise differential invariants, many of which arise from the Lie bracket. Indeed, we can apply the Lie bracket to two vector fields tangent to a distribution ξ and see whether the resulting vector field is tangent as well. If this is always true, we say that ξ is *involutive*. The celebrated theorem by Frobenius says that ξ is involutive if and only if it defines a foliation. We can therefore use the Lie bracket to construct differential invariants that measure the non-involutivity of ξ. These are called *curvatures*.

As topologists, we are interested in the following question: "what is the homotopy type of the space of distributions with given curvatures?". That is, we fix (some of) the differential invariants and we try to see whether we can understand all distributions with such invariants.

A particular family of techniques that has proven to be incredibly fruitful in this context is the *h-principle* [3,4]. I will introduce some of the key ideas behind the h-principle philosophy to prove M. Gromov's classical result on the classification of distributions in open manifolds. We will then turn our attention to close manifolds and go over Y. Eliashberg's result on contact structures in closed 3-manifolds [2]. If time allows I will explain some more recent developments in Contact Topology [5] and Engel Topology [1,6].

This course will hopefully motivate the attendees to look a bit further into this beautiful topic. I will only be able to scratch the surface and therefore many interesting aspects I won't be able to discuss. In particular, I won't get into the rigidity aspects of Contact/Symplectic Topology (holomorphic curves, generating functions, or microlocal sheaves), nor into the more geometric viewpoint regarding distributions (studying local models *à la* Cartan, or looking at energy functionals and their extremals, to name a few). Many different and very active areas of mathematics converge here.

References

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From THU to SAT

Geometric hydrodynamics

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In 1757 Leonhard Euler formulated the partial differential equations governing the motion of an incompressible inviscid fluid—the Euler equations—that every undergraduate in mechanics study today. In 1901, motivated by the dynamics of rotating rigid bodies in liquids, Henry Poincaré wrote down a system of ordinary differential equations describing mechanical systems evolving on finite-dimensional Lie groups. Then, in 1966, Vladimir Arnold made an extraordinary discovery connecting the work of Euler and Poincaré: the Euler equations can be interpreted as a Riemannian geodesic equation on the infinite-dimensional 'Lie group' of volume preserving diffeomorphisms. Arnold's discovery gave rise to a new branch of mathematics today called *geometric hydrodynamics* (or sometimes *topological hydrodynamics*). It turned out that, not only the Euler equations, but many PDE in mathematical physics take the form of geodesic equations on diffeomorphism groups; examples include the KdV, Camassa–Holm, Landau–Lifschitz, and magnetohydrodynamic equations. The geometric insights provided by Arnold's approach yield, for example, a better understanding of stability (by studying sectional curvature) and existence and uniqueness of solutions (by proving smoothness of the geodesic spray in the category of infinite-dimensional Banach manifolds).

The purpuse of my lectures is to give a full account of Arnold's beautiful discovery. If time permits I shall also discuss fruitful connections to other fields, such as optimal mass transport and quantum mechanics.

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The fictitious control method for the internal controllability of underactuated coupled systems of PDEs

From THU to SAT

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Studying the controllability for linear or non-linear coupled systems has been an intensive subject of interest these last years. The main issue is to try to control many equations with less controls than equations, with the hope that one can act indirectly on the equations that are not directly controlled thanks to the coupling terms. In the case of internal controls, one way to prove this kind of property is the use of what is called the "fictitious control method", which was first introduced in [1] in the context of PDEs. This method can be roughly decomposed into two steps:

1. Firstly, control the system with a control acting on each equation. This is in general simpler than the original underactuated problem and may be performed by using classical tools.

2. Secondly, try to find a way to get rid of the control that should not appear in order to obtain the desired results. This can be done using the notion of "algebraic solvability", which relies on a particular construction of a solution for a related underdetermined linear partial differential equations with a source term. This strategy has been introduced in [2] in the context of PDEs.

After having explained the method and highlighted its advantages and drawbacks, I will provide some examples of applications, coming from [3–5].

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CONTRIBUTED TALKS

Lie symmetries for a multi-component NLS Equation in $2+1$ dimensions

FRI
15:30

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Invariance of a differential equation under a set of transformations is equivalent to the existence of symmetries. The study of symmetries represents a fundamental aspect related to the analysis of integrability of differential equations, since this invariance property may be used to achieve partial or complete integration of such equations [8].

The basis of the theory of Lie symmetries lies in the invariance of differential equations under one-parameter transformations of their variables, [7, 8]. These transformations form a local Lie group of transformations, which depend on a continuous parameter, and project any solution of the equation into another solution. Besides, a standard method to find solutions can be implemented by using Lie symmetries: each symmetry leads to a similarity reduction for the PDE which allows to reduce by one the number of independent variables.

Furthermore, as it is well known, a PDE is considered integrable when it can be derived through the Lax equation associated to a spectral problem. Lie symmetries for PDEs are very popular in literature, however, Lie symmetries for Lax pairs are much less frequent [4]. Nevertheless, the determination of the symmetries of the Lax pair has the benefit that the reduction associated to each symmetry of the Lax pair provides not only the reduction of the fields, but also the reductions of the eigenfunctions and the spectral parameter.

This talk is based on the work [1] done by authors. This paper is devoted to the study of an integrable $(2+1)$-dimensional multi-component nonlinear Schrödinger equation

$$\begin{aligned} &i\vec{\alpha}_t + \vec{\alpha}_{xx} + 2\, m_x\, \vec{\alpha} = 0, \quad -i\vec{\alpha}^\dagger_t + \vec{\alpha}^\dagger_{xx} + 2\, m_x\, \vec{\alpha}^\dagger = 0, \\ &\left(m_y + \vec{\alpha}\,\vec{\alpha}^\dagger\right)_x = 0, \end{aligned} \tag{1}$$

where $\vec{\alpha}(x,y,t) = (\alpha_1(x,y,t),\ \alpha_2(x,y,t))^\intercal$ and $\vec{\alpha}^\dagger$ is the complex conjugate of $\vec{\alpha}$. $m(x,y,t)$ is a real scalar function related to the probability density $\vec{\alpha}\,\vec{\alpha}^\dagger$.

The reduction $x = y$ of (1) yields the Manakov system, which is often also called vector NLS system [6]. Integrability properties of this Manakov system are described in [5, 9], and different generalizations of this system and their solutions have been studied in [2, 3].

The associated linear problem, [1], can be written by means of the following three-component Lax pair (and their complex conjugates), with $\vec{\Psi}(x,y,t) = (\psi(x,y,t),\ \chi(x,y,t),\ \rho(x,y,t))^{\intercal}$,

$$\begin{aligned} \psi_y &= -\,\alpha_1^{\dagger}\,\chi - \,\alpha_2^{\dagger}\,\rho, & \psi_t &= -\psi_{xx} - 2\,m_x\,\psi \\ \chi_x &= -\,\alpha_1\,\psi, & \chi_t &= -\,(\alpha_1)_x\,\psi + \alpha_1\,\psi_x \\ \rho_x &= -\,\alpha_2\,\psi, & \rho_t &= -\,(\alpha_2)_x\,\psi + \alpha_2\,\psi_x \end{aligned} \tag{2}$$

The Lie point symmetries of the system (1) and its Lax pair (2) have been determined. The resulting symmetries include nine arbitrary functions of the independent variables and a single arbitrary constant, which plays the role of the spectral parameter when the spectral problem is reduced to $1+1$ dimensions.

The commutation relations among the generators associated to each symmetry have been widely analyzed. Although the set of symmetries does not form a Lie algebra, these relations are consistent and closed. Eventually, we could define the Lie algebra associated to particular selections for the arbitrary functions.

Three non-trivial reductions to $1+1$ dimensions have been identified. The reduced equations and the reduced spectral problem have been simultaneously obtained. It is important to notice that the spectral parameter arises naturally in the process of constructing the reductions, due to the symmetry procedure itself.

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Collective symplectic integrators on $S_2^n \times T^*\mathbb{R}^m$

THU
17:30

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Symplectic integrators for the numerical treatment of Hamiltonian mechanics are a cornerstone of geometric numerical integration. For non-flat geometries, the standard methods do not apply. In some cases *Collective symplectic integrators* apply. In this talk we study such integrators for a symplectic space $S_2^n \times T^*\mathbb{R}^m$. This space arises in an atomistic model of magnetic materials, the spin-lattice-electron model.

THU
18:30

Poisson–Hopf algebras in Lie–Hamilton systems

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We merge quantum algebras with Lie systems in order to establish a new formalism, say Poisson–Hopf algebra deformations of Lie systems. Our procedure can be applied to those Lie systems endowed with a symplectic structure, the so called Lie–Hamilton systems. This is a general approach since it can be applied to any quantum deformation, any underlying manifold and any dimension. One of its main features is that, under quantum deformation, Lie systems are promoted to involutive distributions. Thus a quantum deformed Lie system has no longer an underlying Vessiot–Guldberg Lie algebra nor a quantum algebra one. However, it keeps a (deformed) Poisson–Hopf algebra structure which enables one to obtain, in an explicit way, the t-independent constants of motion from quantum deformed Casimir invariants which can be useful in a further construction of the corresponding deformed superposition rules. Moreover, we illustrate our general approach by considering the non-standard quantum deformation of sl(2) applied to well-known Lie systems, such as the oscillator problem or Milne–Pinney equation and several types of Riccati equations.

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Drinfel'd doubles and coisotropic spacetimes

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17:00

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In this talk we will present and explicitly construct all Drinfeld double (DD) decompositions of the Poincaré Lie group G. We show that there exist eight such non-isomorphic DD structures [1]. Each of these structures define a Poisson-Lie group (G, Π) with associated Lie bialgebra $(\mathfrak{g}, \delta)$.

The coisotropy condition for the cocommutator δ, i.e. $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$, where $\mathfrak{h} = \mathrm{Lie}(H)$ is the Lorentz algebra, is a necessary and sufficient condition for $(\mathcal{M}, \pi)$ to be a Poisson homogeneous space (PHS), where $\mathcal{M} = G/H$ is Minkowski spacetime and π is the canonical projection of Π. Some of these PHS verify the stronger condition $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$, which are less numerous but can be more easily promoted to quantum homogeneous spaces (see [2–4] and references therein).

We will explicitly construct these coisotropic PHS arising as the semiclassical limit of non-commutative spacetimes, which are expected to be one of the footprints of quantum gravity effects at the Planck scale. In fact, we will see that our procedure gives the full non-commutative spacetimes in all but one case [1]. In the last part of the talk we will also present the full classification of Poisson-Lie structures for the Poincaré Lie group G, based on the classification of r-matrices given in [5] and we will identify which of them come from a DD structure.

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THU
15:30

Stabilization of slow-fast systems at non-hyperbolic points

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Slow-fast systems, also known as singularly perturbed ordinary differential equations, are often used to model phenomena in two or more time scales. Examples of such phenomena include: robots with flexible joints, power networks, neuronal dynamics, population dynamics, biochemical reactions, among many others. A convenient property that slow-fast systems may have comes from the fact that, under certain conditions concerning the stability of equilibrium points of the fast dynamics, the whole system can be decomposed into reduced subsystems which together provide a good approximation of the slow-fast dynamics. Taking advantage of the aforementioned decomposition, a classical way to control slow-fast systems is the composite control technique, which consists on the design of sub-controllers for the slow and for the fast subsystems independently. Then, provided that some technicalities are met, the sum of such sub-controllers provides a controller for the slow-fast system.

Although the composite control method is powerful and has had many applications, it fails at non-hyperbolic points of the fast dynamics, which we call singularities. Near singularities, a clear time scale separation is not possible, and usually the trajectories of the slow-fast system exhibit jumps. In this talk we first briefly review the classical composite control method, and then show some examples where such method fails. Next, we present a novel controller that allows the stabilization of singularities of a class of slow-fast control systems. The main ingredient for the design of the controller is the use of geometric desingularization via the blow-up method. Finally, we digress on possible extensions and potential applicability of the theory.

Result

A slow-fast control system (SFCS) is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x,z,u,\varepsilon) \qquad (3) \\ \varepsilon\dot{z} &= g(x,z,u,\varepsilon), \qquad (4) \end{aligned}$$

where $x \in \mathbb{R}^{n_s}$ (slow variable), $z \in \mathbb{R}^{n_f}$ (fast variable), $u \in \mathbb{R}^m$ is a control input, f and g are sufficiently smooth functions, and the independent variable is the slow time t. One can also define a new time parameter $\tau = \frac{t}{\varepsilon}$ called the fast time, and then the system (3)-(4) is rewritten as

$$\begin{aligned} x' &= \varepsilon f(x,z,u,\varepsilon) \quad (5)\\ z' &= g(x,z,u,\varepsilon), \quad (6)\end{aligned}$$

where the prime $'$ denotes derivative with respect to τ. Note that equations (3)-(4) and (5)-(6) are equivalent as long as $\varepsilon > 0$.

Definition 1 (Normal hyperbolicity). *A point $s \in \mathcal{S}$ is called hyperbolic if it is a hyperbolic equilibrium point of the reduced vector field $z' = g(x,z,0)$ of the Layer equation. The manifold $\mathcal{S}$ is called normally hyperbolic (NH) if every point $s \in \mathcal{S}$ is hyperbolic. A point that fails to be hyperbolic is called* non-hyperbolic.

Our main result is as follows (full details, the proof, and a couple of examples appear on [1]):

Theorem 1. *Consider the SFCS*

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) + B(x,z,\varepsilon)u(x,z,\varepsilon) \quad (7)\\ \varepsilon\dot{z} &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + H(x,z,\varepsilon), \quad (8)\end{aligned}$$

where B is invertible near the origin and $H(x,z,\varepsilon)$ denotes higher order terms. Let us denote the i-th component of the vector Bu as $(Bu)_i$. Suppose the controller u is designed such as

$$\begin{aligned} (Bu)_1 &= -A_1 + \varepsilon^{\frac{-1}{2k-1}}(1+c_0c_1)z + \varepsilon^{\frac{-k}{2k-1}} \sum_{i=2}^{k-1} c_i x_i z^{i-1}\\ &\quad + \varepsilon^{-1}\left(\frac{\partial G_k}{\partial z} - \varepsilon^{\frac{k-1}{2k-1}}(c_0+c_1)\right) G_k\\ (Bu)_i &= -A_i - c_i \varepsilon^{\frac{-k}{2k-1}} x_i,\\ (Bu)_j &= -A_j - c_j \varepsilon^{\frac{-k}{2k-1}} x_j,\end{aligned}$$

where all constants c_0, c_1, c_i, c_j are positive with $c_i \ll c_1$ for $i = 0, 2, \dots, k-1$, $j = k, \dots, n_s$. Then the origin $(x,z) = (0,0) \in \mathbb{R}^{n_s} \times \mathbb{R}$ is rendered locally asymptotically stable for $\varepsilon > 0$ sufficiently small.

Remark: note that the origin is a non-hyperbolic point of the layer equation of the system (7)-(8). In fact, at the origin we have $g(0) = \frac{\partial g}{\partial z}(0) = \cdots = \frac{\partial^{k-1} g}{\partial z^{k-1}}(0) = 0$ and $\frac{\partial^k g}{\partial z^k}(0) \neq 0$.

References

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THU
16:30

Groupoids and Distributions: A new way to deal with non-uniform material bodies

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A groupoid, called *material groupoid*, is associated in a natural way over a general non uniform body (see [1–3]). The *material distribution* is introduced due to the (possible) lack of differentiability of the material groupoid (see [4, 5]). Thus, the inclusion of these new objects in the theory of material bodies opens the possibily of studying non-uniform bodies. As an example, the material distribution and its associated singular foliation result in a rigorous and unique subdivision of the material body into strictly smoothly uniform sub-bodies, laminates, filaments and isolated points. Furthermore, the material distribution permits us to present a "measure" of uniformity of a simple body as well as more general definitions of homogeneity for non-uniform bodies.

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From (quantum) geometry to spectral theory

FRI
18:00

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The area operator, that appears in the context of Loop Quantum Gravity, has as eigenvalues all possible numbers of the form

$$\sum_{i=1}^{n} \sqrt{n_i(n_i+2)}$$

for some natural numbers n_i. Several approximate methods have been proposed over the years to study the distribution of these eigenvalues. They rely on approximations to get rid of the square root and known results about integer partitions, in particular, the classic asymptotic estimates due to Hardy, Ramanujan, and Rademacher. The main problem with these approaches is that different approximations lead to different results and, hence, are not conclusive. In this talk, I will present a method that we developed, based on Laplace transforms, that provides a very accurate solution to this problem. The representation that we get is valid for any area and can be used, in particular, to obtain its asymptotics in the large area limit.

FRI
17:30

A Homotopy co-momentum Map in Hydrodynamics

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In this talk, based on joint work with M. Spera [1], we investigate some connections between multisymplectic geometry and hydrodynamics.
After a brief review of the basic definitions of *multisymplectic manifold* and *Lie-∞ algebra of observables* [2], we recall the notion of an *Homotopy co-momentum map* [3] and realize an explicit construction of such object in the case of $\mathbb{R}^3$ which is relevant to hydrodynamics. In this way, we are able to reinterpret the so-called *Rasetti-Regge currents* [4], introduced in the contest of vortex dynamics, as momenta associated to the vorticity.
Time permitting, we shall discuss a generalization of the above construction in the case of *perfect fluid* on compact, oriented Riemannian manifold satisfying appropriate cohomological conditions.
The former construction finds an application in knot theory starting from the observation that n-links can be related to suitable *conserved quantities* [5].

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A Necessary Condition for Local Controllability

FRI
18:00

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Local controllability around an equilibrium is an important notion within control theory. Necessary or sufficient conditions for small-time local controllability (STLC) have been much investigated in the last decades. Some powerful sufficient conditions have been stated; however, most necessary conditions for STLC are more specific and deal with scalar-input control systems, including the classical result from Sussmann in [3].

The purpose of this talk is to extend this necessary condition to a particular class of systems with two controls, in which the field associated to the second control vanishes at the equilibrium point. In this case, the second control may allow better local controllability in some sense, provided the control vector fields verify another Lie bracket hypothesis.

Let f_0, f_1 be analytic vector fields on $\mathbf{R}^n$. Consider the control-affine system with scalar-input control

$$\dot{z} = f_0(z) + u_1(t) f_1(z), \tag{9}$$

with $f_0(0) = 0$ (meaning that $(0,0)$ is an equilibrium point for the system).

Definition 1 (STLC)**.** *The control system* (9) *is* STLC at $(0,0)$ *if, for every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that, for every* z_0, z_1 *in the ball centered at* 0 *with radius* η*, there exists a control* $u(\cdot)$ *in* $L^\infty([0,\varepsilon])$ *such that the solution of the control system* $z(\cdot) : [0,\varepsilon] \to \mathbf{R}^n$ *of* (9) *satisfies* $z(0) = z_0$*,* $z(\varepsilon) = z_1$*, and*

$$\|u\|_{L^\infty} \leqslant \varepsilon\,.$$

Note that this definition, used by Coron in [1], requires smallness both in time and in control. Nevertheless, another definition that only requires boundedness (and not smallness) of the control can be found in the works of Hermes and Sussmann [3] among others. This second definition, while not equivalent to the first one, is sometimes called STLC as well. In order to avoid the confusion, we will call it α-STLC:

Definition 2 (α-STLC)**.** *Let $\alpha \geqslant 0$. The control system* (9) *is* α-STLC at $(0,0)$ *if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in the ball centered at* 0 *with radius η, there exists a control $u(\cdot)$ in $L^\infty([0,\varepsilon])$ such that the solution of the control system $z(\cdot) : [0,\varepsilon] \to \mathbf{R}^n$ of* (9) *satisfies $z(0) = z_0$, $z(\varepsilon) = z_1$, and*

$$\|u\|_{L^\infty} \leqslant \alpha + \varepsilon\,.$$

Let us call S_1 the subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ spanned by all the Lie brackets of f_0, f_1 containing f_1 at most one time, and $S_1(0)$ the subspace of $\mathbf{R}^n$ spanned by the value at 0 of the elements of S_1. The classical result from Sussmann states that the value at 0 of a certain bracket in which f_1 appears two times needs to belong to $S_1(0)$ for any STLC to hold.

Theorem 1. *Assume that $[f_1,[f_0,f_1]](0) \notin S_1(0)$. Then,* (9) *is not STLC(α) for any α.*

Our result proposes an extension of this result to the case of systems with a second scalar control u_2 against a vector field f_2 that vanishes at the equilibrium point. Consider the new system

$$\dot{z} = f_0(z) + u_1(t) f_1(z) + u_2(t) f_2(z) \tag{10}$$

with $f_0(0) = 0$ and $f_2(0) = 0$.

Similarly to the previous case, we call R_1 the subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ spanned by all the Lie brackets of f_0, f_1, f_2 containing f_1 at most one time, and $R_1(0)$ the subspace of $\mathbf{R}^n$ spanned by the value at 0 of the elements of S_1.

Theorem 2. *Assume that $[f_1,[f_0,f_1]](0) \notin R_1(0)$. Then,*

1. *if $[f_1,[f_0,f_1]](0) \in \mathrm{Span}(R_1(0), [f_1,[f_2,f_1]](0))$,* (10) *is not STLC.*
2. *else,* (10) *is not STLC(α) for any α.*

Remark 1. *Note how adding the second control u_2 changes the controllability level of the system in the first case. Provided that the right Lie brackets are linked at 0, the system is not STLC, but might be α-STLC for some $\alpha > 0$, which is never the case without a second control.*

The proof, which we will develop during the presentation, is based on the representation of trajectories through the Chen-Fliess series (see again [3]).

We will give an illustration of this result on the equations describing the 2D dynamics of a magnetic-driven elastic micro-swimmer made of two rigid links, studied in [2]. The equations of motions may be written as a control system with two control inputs (the two components of the magnetic field in the motion plane), that fits the hypothesis $f_0(0) = f_2(0) = 0$. The theorem ensures that the micro-swimmer is not STLC, but it is shown in [2] that it is α-STLC, with α depending on the physical parameters of the system.

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Numerical optimal control of nonholonomic systems

THU
17:30

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In order to solve optimal control problems, standard calculus of variations can be applied to a Hamilton-Pontryagin action [1] to obtain a set of equations conforming, together with initial and terminal conditions, the necessary conditions for optimality [2]. Discretization of this principle allows us to obtain numerical methods naturally suited for these problems with good behaviour and geometric properties [3] [4].

Nonholonomic mechanical systems are a type of constrained systems whose equations of motion cannot be derived from a standard variational principle [5] [6]. These equations can be thought of as a special case of forced system and it is very common to default to this interpretation when dealing with optimal control problems for these kinds of systems. Unfortunately when we discretize using this interpretation, the constraint is generally not preserved.

In this talk I will show our latest strides in achieving a correct discretization of the Hamilton-Pontryagin action for the optimal control problem of nonholonomic mechanical systems with exact constraint preservation.

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SAT
12:30

Geometrical framework of gauge theories

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In classical field theory fields are represented by sections of the fibered bundle $E \to M$. The configuration space of the theory is a bundle of first jets $\mathsf{J}^1 E$ same as a tangent bundle $\mathsf{T}M$ is in mechanics. However, there exist several geometrical formulations of classical field theory where most of them are usually associated with some extended symplectic structures (multisymplectic structures, k-symplectic structures, k-cosymplectic structures etc.) on manifolds. One of the approach based on the so-called multisymplectic structures depends on the construction of Poincare-Cartan forms on the Lagrangian side and Hamilton-De Donder-Weyl equations on the Hamiltonian one.

This very general picture may be usually simplified when we consider some particular case of field theory. For instance, in a case of gauge theories the bundle of fields is a bundle of connections in a principal bundle $P \to M$. In my talk I will present main geometrical aspects of gauge (especially Yang-Mills type) field theories and its relations to the general formalism.

POSTER SESSION

THU
From
16:00
to
17:00

Dynamics of pendula hanging from a string

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We present the dynamics of a hybrid mechanical system composed of a homogeneous flexible and elastic string, with fixed ends, on which are suspended two identical pendula. In our model we include a dissipative contribution due to internal damping within the string, modelling it as viscoelastic friction.

The aim of this study is twofold: we want on the one hand to analyse how the continuous nature of the coupling between the pendula affects their small oscillations, and on the other to investigate the emergence of synchronization patterns and to provide an interpretation of synchronization phenomena in mechanical systems as an outcome of damping contributions affecting the normal modes of oscillation.

The work, making use of the Lagrangian formalism, provides a linear analysis based on the determination of the damped normal modes of oscillation. Hence, we investigate beats and synchronization by means of a numerical study as the parameters that characterize the system change.

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The reveal of minimal ruled surfaces using control theory

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We seek minimal surfaces among the family of ruled surfaces in the Euclidean space. Such surfaces are generated by a straight line, called generatrix, that moves along a curve, the directrix. We approach this problem using the techniques from control theory. In the literature the minimal revolution surface is often obtained as an optimal control problem because of the peculiarity of the parameterization that reduces the problem to an ordinary differential equation. Nothing similar has been published for other families of minimal surfaces because partial differential equations cannot be avoided. This is why we use the k-symplectic formalism, which allows us to solve the general optimal control problem by means of the Hamilton-De Donder-Weyl equation. Applying it to minimize the area of a ruled surface subject to a dynamics associated with the surface we recover the plane and the helicoid as candidates to be minimizers. These two surfaces are the only minimal ruled surfaces, as known from the geometrical characterization of zero mean curvature for the minimal surfaces.

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Nonlocal symmetries of Gibbons–Tsarev equation

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We consider the 3D integrable equation discovered independently by Mikhalev and Pavlov, $u_{yy} = u_{tx} + u_y uxx - u_x u_{xy}$ and its 2D reduction $u_{yy} = (u_y + y)u_{xx} - u_x u_{xy} - 2$ which is equivalent to the Gibbons-Tsarev equation. These equations play an important role in the theory of integrable systems, and the Gibbons–Tsarev equation is also related to the theory of conformal maps. We will present new results on the differential coverings and nonlocal symmetries of the equations in question. For details please see the paper [1].

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[3]The research of PH was supported by the Specific Research grant SGS/6/2017 of the Silesian University in Opava.

Geometric description of quantum states: Markovian dynamics and control of open quantum systems

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Differential geometry allows us to analyse the underlying structures in the space of pure and mixed states of quantum systems. I will describe the geometric objects that arise naturally from the properties of quantum systems, namely a Poisson and a symmetric tensor field which define a specific stratification of the manifold of states. This geometric structure is a powerful framework for the study of dissipation, Markovian dynamics, decoherence and optimal control problems in open quantum systems.

References

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On the Controllability of Rolling Pseudo-Hyperbolic Space - A Constructive Proof

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This poster presents some recent results [2] concerning a constructive proof for complete controllability of the rolling motion of a 2-dimensional pseudo-hyperbolic space, with index zero, over the affine space associated with the tangent space at a point. This rolling motion is assumed to have the constraints of no-twist and no-slip (pure rolling).

[4]Based in joint work with J. F. Cariñena, J. Clemente-Gallardo and G. Marmo.

1 Establishing the problem

Consider the matrix $J_\nu = \mathrm{diag}(-I_\nu, I_{n-\nu})$, with $0 \leq \nu \leq n$. The formula $\langle u, w\rangle_\nu = u^\top J_\nu w$ defines a scalar product on the vector space $\mathbb{R}^n$. Equipping each $T_p\mathbb{R}^n \equiv \mathbb{R}^n$ with $\langle \cdot, \cdot\rangle_\nu$, the manifold $\mathbb{R}^n$ becomes a pseudo-Riemannian manifold, which we denote by $\mathbb{R}^n_\nu$.

Associated with J_ν, one also defines the group $\mathrm{O}_\nu(n) = \{R \in GL(n, \mathbb{R}) : R^\top J_\nu R = J_\nu\}$. The identity component of $\mathrm{O}_\nu(n)$ is denoted by $\mathrm{SO}^+_\nu(n)$. Since each isometry of $\mathbb{R}^n_\nu$ has a unique expression as $x \mapsto Rx + s$, with $R \in \mathrm{O}_\nu(n)$ and $s \in \mathbb{R}^n$, we identify the isometry group of $\mathbb{R}^n_\nu$ with $\mathrm{O}_\nu(n) \rtimes \mathbb{R}^n$. Its maximal connected subgroup is $\mathrm{SO}^+_\nu(n) \rtimes \mathbb{R}^n$.

The *pseudo-hyperbolic space* in $\mathbb{R}^{n+1}_{\nu+1}$ is the hyper-quadric, with index ν, dimension n and radius r, defined by $H^n_\nu(r) = \{p \in \mathbb{R}^{n+1}_{\nu+1} : \langle p, p\rangle_{\nu+1} = -r^2\}$. The *affine tangent space* to $H^n_\nu(r)$ at a point p_0 is $T^{\mathrm{aff}}_{p_0} H^n_\nu(r) = \{p_0 + v : v \in T_{p_0} H^n_\nu(r)\}$.

The main result about the rolling motion of $H^n_\nu(r)$ over $T^{\mathrm{aff}}_{p_0} H^n_\nu(r)$ was proved in [1] and is presented next. Let p_0 be a point in $H^n_\nu(r)$ and $t \in [t_0, t_1] \mapsto u(t) \in \mathbb{R}^{n+1}_{\nu+1}$ a (piecewise) smooth function satisfying $\langle u(t), p_0\rangle_{\nu+1} = 0$. If $(R(t), s(t)) \in \mathrm{SO}^+_{\nu+1}(n+1) \rtimes \mathbb{R}^{n+1}$ is the solution-curve of the system $\dot{R}(t) = R(t)\left(-u(t)p_0^\top + p_0 u^\top(t)\right) J_{\nu+1} \ \wedge \ \dot{s}(t) = r^2 u(t)$ $(*)$, satisfying $(R(t_0), s(t_0)) = (R_0, s_0)$, with $s_0 \in T_{p_0} H^n_\nu(r)$, then $X(t) = (R^{-1}(t), s(t))$ defines a rolling map of $H^n_\nu(r)$ over $T^{\mathrm{aff}}_{p_0} H^n_\nu(r)$, without slipping or twisting.

The kinematic equations $(*)$ can be seen as a control system, and we have proved in [1] that it is controllable on the Lie group $G = \{(R, s) : R \in \mathrm{SO}^+_{\nu+1}(n+1), s \in T_{p_0} H^n_\nu(r)\}$, if $n \geq 2$. However, the proof of the controllability property presented in [1] is not constructive, i.e., does not specify how to reach a configuration from another. Thus, it makes sense to present a constructive proof of controllability, which is precisely the purpose of this work.

We will consider the case $n = 2$, $\nu = 0$ and, without loss of generality, $p_0 = (1, 0, 0)$. A constructive proof of the controllability property corresponds to showing how it is possible to replace the forbidden "twists" and "slips" by pure rolling motions, to obtain the same effect. Therefore, the key issues of this work can be formulated as follows: *i*) how to generate a twist or a sliding twist associated with any given angle?, *ii*) how to generate a pure slip associated with any given displacement?

2 Answers to the essential questions

Consider an arbitrary angle $\theta \in \mathbb{R}$. Next we present a rolling motion that generates a sliding twist of $H^2_0(1)$ over $T^{\mathrm{aff}}_{p_0} H^2_0(1)$, correspondent to a rotation θ around the x-axis. For that, we start by choosing any non-zero auxiliary value φ, having opposite sign to the given angle θ.

Algorithm 2.1. *(to generate a sliding twist)*

Step 1. *The pseudo-hyperbolic space rolls over the vector* $u = (0, \varphi, 0)$. *Control:* $u(t) = u$.

Step 2. *The pseudo-hyperbolic space rolls over a line segment parallel to the* z*-axis, with length* $-\theta/\sinh(\varphi)$. *Control:* $u(t) = (0, 0, -1)$.

Step 3. *The pseudo-hyperbolic space rolls back over a line segment parallel to the* y*-axis, with length* $|\varphi|$. *Control:* $u(t) = (0, -\varphi, 0)$.

Step 4. *The pseudo-hyperbolic space rolls on a circunference centred at* $(0, \coth(\varphi), \frac{\theta}{\sinh(\varphi)})$, *with radius equal to* $|\coth(\varphi)|$, *describing an angle* θ.

Control: $u(t) = \cosh(\varphi)(0, \sin(\sinh(\varphi)(t - 2 - \frac{\theta}{\sinh(\varphi)})), \cos(\cosh(\varphi)(t - 2 - \frac{\theta}{\sinh(\varphi)})))$. □

Consider an arbitrary vector $s \in T_{p_0}H_0^2(1)$. We now present a rolling motion that generates a pure slip of $H_0^2(1)$ over $T_{p_0}^{\text{aff}}H_0^2(1)$, correspondent to the displacement s. Set $\varphi \in I\!R^-$ and denote by $T_\varphi(\theta)$ the translation operated on $H_0^2(1)$ when a sliding twist, with rotation angle θ and auxiliar value φ, is generated.

Algorithm 2.2. *(to generate a pure slip)*
Step 1. Perform the rolling motion along the vector $u = -T_\varphi(\pi) + \frac{1}{2}s$. Control: $u(t) = u$.
Step 2. Perform the rolling that generates the sliding twist of angle $\theta = \pi$, using the auxiliary value φ. (At the end the "midpoint" of the vector s is reached.)
Step 3. Repeat the first step, that is, roll the pseudo-hyperbolic space again along the vector u, applied at the midpoint of s.
Step 4: Repeat the second step, to generate another sliding twist of angle π.

□

The figure below illustrates the two previous algorithms.

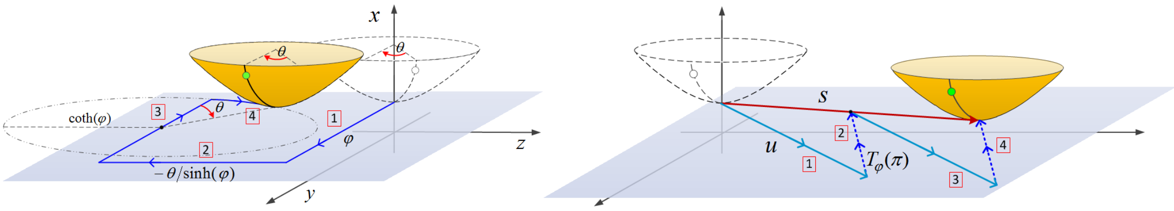

Figure 1: On the left a sliding twist is generated. On the right is generated a pure slip.

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Rolling Grassmannians – Constructive Proof of Controllability

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The poster will address controllability properties of the kinematic equations describing rolling motions of Grassmann manifolds (Grassmannians) over their affine tangent space at a point. These rolling motions, which are assumed to have nonholonomic constraints of no-slip and no-twist, result from the action of a certain Lie group on the vector space of symmetric matrices where the rolling manifold and the static manifold are embedded.

The basis for this work is the article [1], where the kinematic equations have been derived. These kinematic equations can be rewritten as a nonlinear control system evolving on a particular connected Lie group and it turns out that the system is controllable. This analytical proof is based on showing that the control vector fields have the bracket generating property, but does not give any insight on how to join two different configurations of the Grassmannian using trajectories of the control system only.

The objective of this poster is to highlight a constructive proof of controllability of the rolling system, by showing how the forbidden motions of twisting and slipping can be accomplished by rolling without breaking the nonholonomic constraints of no-slip and no-twist. This geometric construction generalizes a similar approach that was accomplished in [2] for the rolling sphere. Details concerning controllability of the rolling Grassmannians and, in particular, this constructive proof can be found in [3].

The study of these problems was motivated by the potential applications of the Grassmann manifold in several engineering areas that deal with sets of images, such as face recognition problems under varying illumination conditions, or reconstruction of planar scenes from multiple views (see, for instance, [4] and [5] for several interesting real problems about this topic).

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Affine Lagrangians in the k-cosymplectic formulation of field theories

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Field theories described by Lagrangians depending on space-time coordinates can be modelled using k-cosymplectic geometry. In particular, theories described by singular Lagrangians are of special interest because of their role in modern physics. Nevertheless there is a problem of consistency of the corresponding PDEs (Euler–Lagrange and Hamilton–de Donder–Weyl), and thus one needs to study the possible constraints arising in the space of solutions. Here we define the concept of k-precosymplectic manifold and show that they possess Darboux coordinates. We also prove the existence of Reeb vector fields for this structure. Then we define the notion of k-precosymplectic Hamiltonian system and develop a geometric constraint algorithm in order to find a constraint submanifold where one can assure the existence of solutions. We are going to apply this algorithm to the case of affine lagrangians in both the Lagrangian and the Hamiltonian formalisms.

Tracking and filtering on SE(2)

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Attitude estimation is a core problem in many robotic systems, such as unmanned aerial and ground vehicles. The configuration space of these systems is properly modelled exploiting the theory of Lie groups. In this paper we propose a second-order-optimal minimum-energy filter on the matrix Lie group SE(2). The mathematics behind it is quite challenging and is not a simple generalization of previous results on Lie groups.

In the last decades many linear and nonlinear, deterministic and stochastic, observers have been proposed in the literature, some of them exploiting the theory of Lie groups to provide the proper mathematical structure for the attitude of a mechanical system [1].

We propose a *second order optimal minimum energy filter*. The filter is based on the results of Mortensen [2] where a methodology of generating progressive realizable approximations of a minimum-energy functional was proposed. The solution is obtained by differentiating the boundary conditions of the associated optimal control problem. It is called *second order* optimal in the sense that it is a truncation of the exact solution that would be an infinite dimensional system. The filter takes the form of a gradient observer coupled with a kind of Riccati differential equation that updates its gain (similarly to the standard Kalman filter).

The theoretical result in [3] is our starting point for the under submission paper [4]. We apply the theorem for a general Lie group to the case of a system modelled on the Lie group SE(2). Such group is very important to model vehicles moving on a plane and so, coupled with the dynamical part, to estimate their pose. However it is worth highlighting that this work is not just a 'straightforward' extension of the result in [3]. The SE(2) case turns out to be particularly complicated since we don't have the Lie algebra isomorphism between $\mathfrak{se}(2)$ and $\mathbb{R}^3$. Contrary to the SO(3) case, we can't identify the bracket operation of the Lie algebra $\mathfrak{se}(2)$ with the classical cross product in $\mathbb{R}^3$.

Our main contribution is then to derive a second order optimal minimum energy filter for SE(2) and to provide all the mathematics for the many technical operations needed to compute it.

We formulate the problem and we recall the explicit formula for the second-order-optimal minimum-energy filter on a general Lie group as stated in [3]. We described the SE(2) case, where we provide the main contribution in a Proposition with proof and all the mathematical details.

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Exact discrete lagrangian for constrained mechanics: an open problem

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The existence of an exact discrete lagrangian function for constrained systems (nonholonomic and vakonomic) is still an open problem in the field of geometric integration (see [1] for the unconstrained case). In the last few decades, an effort has been made to introduce geometric numerical methods, such as variational integrators, which preserve geometric structure. In the case of variational integrators, we discretize the lagrangian function to which we apply a discrete variational principle to obtain the discrete-time equations of motion, whose solutions are sequences of points which approximate the solution for the continuous-time problem.

In this talk we discuss constrained mechanics. After exposing the corresponding discrete descriptions (cf. [6], [8] or [7] for nonholonomic, [2] for vakonomic and [3], [5] for an introduction to constrained systems), we introduce the problem of finding an exact discrete lagrangian function for constrained mechanical systems. We will unveil the exact discrete space where

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nonholonomic dynamics takes place and explicitly define the corresponding exact retraction, using a nonholonomic connection defined in [4].

The discovery of the nonholonomic and vakonomic discrete exact lagrangian function will make an advance to the study of error analysis of numerical methods. For instance, an exact discrete constrained lagrangian function would have many applications on optimal control.

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Lie–Poisson simulation of the dynamics of point vortices on a rotating sphere coupled with a background field

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One of the major unsolved problems of the Euler equations on a sphere is the long-time behaviour of an inviscid fluid when a certain initial vorticity is given [4], [6]. The

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biggest challenge in understanding and simulating this system is due to the presence of an infinite number of first integrals (Casimir functions), which however don't provide complete integrability [7]. Moreover, the Hamiltonian nature of the equations suggests that much geometry is involved in them. They are in fact a Lie–Poisson system on $sdiff^*(\mathbb{S}^2)$ (the dual of the Lie algebra of divergence free vector fields) [1]. Special solutions to the Euler equations come from the fact that on $sdiff^*(\mathbb{S}^2)$ there exist non trivial finite dimensional coadjoint orbits, called *point vortices* [5]. However, this orbits have physical interest only when the sphere is non-rotating. When this is not true the Euler equations become a coupled system of equations of a singular field of point vortices and a smooth continuous background vorticity [3].

Starting from [2], [8], we present an approximation of the coupled model based on the quantization of Kähler manifolds, which keeps the Hamiltonian Lie-Poisson structure of the equations. Moreover, with the techniques of the geometric integration, we provide a Lie-Poisson numerical scheme to solve the quantized model, preserving up to roundoff precision the discrete Casimirs and, up to the order of the method, the Hamiltonian. The conservation of the quantized first integrals provides a deep insight in the nature of the Euler equations and a better qualitative simulation of them. Furthermore, our numerical scheme provides a useful tool in studying the still unknown persistence of relative equilibria of point vortices passing from a non-rotating to a rotating sphere.

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