Stabilization of slow-fast systems at non-hyperbolic points

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Slow-fast systems, also known as singularly perturbed ordinary differential equations, are often used to model phenomena in two or more time scales. Examples of such phenomena include: robots with flexible joints, power networks, neuronal dynamics, population dynamics, biochemical reactions, among many others. A convenient property that slow-fast systems may have comes from the fact that, under certain conditions concerning the stability of equilibrium points of the fast dynamics, the whole system can be decomposed into reduced subsystems which together provide a good approximation of the slow-fast dynamics. Taking advantage of the aforementioned decomposition, a classical way to control slow-fast systems is the composite control technique, which consists on the design of sub-controllers for the slow and for the fast subsystems independently. Then, provided that some technicalities are met, the sum of such sub-controllers provides a controller for the slow-fast system.

Although the composite control method is powerful and has had many applications, it fails at non-hyperbolic points of the fast dynamics, which we call singularities. Near singularities, a clear time scale separation is not possible, and usually the trajectories of the slow-fast system exhibit jumps. In this talk we first briefly review the classical composite control method, and then show some examples where such method fails. Next, we present a novel controller that allows the stabilization of singularities of a class of slow-fast control systems. The main ingredient for the design of the controller is the use of geometric desingularization via the blow-up method. Finally, we digress on possible extensions and potential applicability of the theory.

Result

A slow-fast control system (SFCS) is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x,z,u,\varepsilon) \\ \varepsilon\dot{z} &= g(x,z,u,\varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_s}$ (slow variable), $z \in \mathbb{R}^{n_f}$ (fast variable), $u \in \mathbb{R}^m$ is a control input, f and g are sufficiently smooth functions, and the independent variable is the slow time t. One can also define a new time parameter $\tau = \frac{t}{\varepsilon}$ called the fast time, and then (1) is rewritten as

$$\begin{aligned} x' &= \varepsilon f(x,z,u,\varepsilon) \\ z' &= g(x,z,u,\varepsilon), \end{aligned} \tag{2}$$

where the prime $'$ denotes derivative with respect to τ. Note that (1) and (2) are equivalent as long as $\varepsilon > 0$.

Definition 1 (Normal hyperbolicity). *A point $s \in \mathcal{S}$ is called hyperbolic if it is a hyperbolic equilibrium point of the reduced vector field $z' = g(x,z,0)$ of the Layer equation. The manifold $\mathcal{S}$ is called normally hyperbolic (NH) if every point $s \in \mathcal{S}$ is hyperbolic. A point that fails to be hyperbolic is called* non-hyperbolic.

Our main result is as follows (full details, the proof, and a couple of examples appear on [1]):

Theorem 1. *Consider the SFCS*

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) + B(x,z,\varepsilon)u(x,z,\varepsilon) \\ \varepsilon\dot{z} &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + H(x,z,\varepsilon), \end{aligned} \tag{3}$$

where B is invertible near the origin and $H(x,z,\varepsilon)$ denotes higher order terms. Let us denote the i-th component of the vector Bu as $(Bu)_i$. Suppose the controller u is designed such as

$$\begin{aligned} (Bu)_1 &= -A_1 + \varepsilon^{\frac{-1}{2k-1}}(1+c_0c_1)z + \varepsilon^{\frac{-k}{2k-1}} \sum_{i=2}^{k-1} c_i x_i z^{i-1} \\ &\quad + \varepsilon^{-1}\left(\frac{\partial G_k}{\partial z} - \varepsilon^{\frac{k-1}{2k-1}}(c_0+c_1)\right) G_k \\ (Bu)_i &= -A_i - c_i\varepsilon^{\frac{-k}{2k-1}} x_i, \\ (Bu)_j &= -A_j - c_j\varepsilon^{\frac{-k}{2k-1}} x_j, \end{aligned} \tag{4}$$

where all constants c_0, c_1, c_i, c_j are positive with $c_i \ll c_1$ for $i = 0, 2, \dots, k-1$, $j = k, \dots, n_s$. Then the origin $(x,z) = (0,0) \in \mathbb{R}^{n_s} \times \mathbb{R}$ is rendered locally asymptotically stable for $\varepsilon > 0$ sufficiently small.

Remark: note that the origin is a non-hyperbolic point of (3). In fact, at the origin we have $g(0) = \frac{\partial g}{\partial z}(0) = \cdots = \frac{\partial^{k-1} g}{\partial z^{k-1}}(0) = 0$ and $\frac{\partial^k g}{\partial z^k}(0) \neq 0$.

References

[1] H. Jardón-Kojakhmetov, J. M. A. Scherpen and D. del Puerto-Flores (2019). *Stabilization of a class of slowfast control systems at non-hyperbolic points*. Automatica.