# On the Controllability of Rolling Pseudo-Hyperbolic Space - A Constructive Proof

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This poster presents some recent results [2] concerning a constructive proof for complete controllability of the rolling motion of a 2-dimensional pseudo-hyperbolic space, with index zero, over the affine space associated with the tangent space at a point. This rolling motion is assumed to have the constraints of no-twist and no-slip (pure rolling).

## 1 Establishing the problem

Consider the matrix $J_\nu = \operatorname{diag}(-I_\nu, I_{n-\nu})$, with $0 \leq \nu \leq n$. The formula $\langle u, w\rangle_\nu = u^\top J_\nu w$ defines a scalar product on the vector space $I\!R^n$. Equipping each $T_p I\!R^n \equiv I\!R^n$ with $\langle \cdot, \cdot \rangle_\nu$, the manifold $I\!R^n$ becomes a pseudo-Riemannian manifold, which we denote by $I\!R^n_\nu$.

Associated with $J_\nu$, one also defines the group $\mathrm{O}_\nu(n) = \left\{R \in GL(n, I\!R) : R^\top J_\nu R = J_\nu\right\}$. The identity component of $\mathrm{O}_\nu(n)$ is denoted by $\mathrm{SO}^+_\nu(n)$. Since each isometry of $I\!R^n_\nu$ has a unique expression as $x \mapsto Rx + s$, with $R \in \mathrm{O}_\nu(n)$ and $s \in I\!R^n$, we identify the isometry group of $I\!R^n_\nu$ with $\mathrm{O}_\nu(n) \rtimes I\!R^n$. Its maximal connected subgroup is $\mathrm{SO}^+_\nu(n) \rtimes I\!R^n$.

The *pseudo-hyperbolic space* in $I\!R^{n+1}_{\nu+1}$ is the hyper-quadric, with index $\nu$, dimension $n$ and radius $r$, defined by $H^n_\nu(r) = \left\{p \in I\!R^{n+1}_{\nu+1} : \langle p, p\rangle_{\nu+1} = -r^2\right\}$. The *affine tangent space* to $H^n_\nu(r)$ at a point $p_0$ is $T^{\mathrm{aff}}_{p_0} H^n_\nu(r) = \{p_0 + v : v \in T_{p_0} H^n_\nu(r)\}$.

The main result about the rolling motion of $H^n_\nu(r)$ over $T^{\mathrm{aff}}_{p_0} H^n_\nu(r)$ was proved in [1] and is presented next. Let $p_0$ be a point in $H^n_\nu(r)$ and $t \in [t_0, t_1] \mapsto u(t) \in I\!R^{n+1}_{\nu+1}$ a (piecewise) smooth function satisfying $\langle u(t), p_0\rangle_{\nu+1} = 0$. If $(R(t), s(t)) \in \mathrm{SO}^+_{\nu+1}(n+1) \rtimes I\!R^{n+1}$ is the solution-curve of the system $\dot R(t) = R(t)\left(-u(t)p_0^\top + p_0 u^\top(t)\right) J_{\nu+1} \ \wedge \ \dot s(t) = r^2 u(t)$ $(*)$, satisfying $(R(t_0), s(t_0)) = (R_0, s_0)$, with $s_0 \in T_{p_0} H^n_\nu(r)$, then $X(t) = (R^{-1}(t), s(t))$ defines a rolling map of $H^n_\nu(r)$ over $T^{\mathrm{aff}}_{p_0} H^n_\nu(r)$, without slipping or twisting.

The kinematic equations $(*)$ can be seen as a control system, and we have proved in [1] that it is controllable on the Lie group $G = \{(R, s) : R \in \mathrm{SO}^+_{\nu+1}(n+1), s \in T_{p_0} H^n_\nu(r)\}$, if $n \geq 2$. However, the proof of the controllability property presented in [1] is not constructive, i.e., does not specify how to reach a configuration from another. Thus, it makes sense to present a constructive proof of controllability, which is precisely the purpose of this work.

We will consider the case $n = 2$, $\nu = 0$ and, without loss of generality, $p_0 = (1, 0, 0)$. A constructive proof of the controllability property corresponds to showing how it is possible to replace the forbidden "twists" and "slips" by pure rolling motions, to obtain the same effect. Therefore, the key issues of this work can be formulated as follows: *i*) how to generate a twist or a sliding twist associated with any given angle?, *ii*) how to generate a pure slip associated with any given displacement?

## 2 Answers to the essential questions

Consider an arbitrary angle $\theta \in I\!R$. Next we present a rolling motion that generates a sliding twist of $H_0^2(1)$ over $T_{p_0}^{\text{aff}} H_0^2(1)$, correspondent to a rotation $\theta$ around the $x$-axis. For that, we start by choosing any non-zero auxiliary value $\varphi$, having opposite sign to the given angle $\theta$.

**Algorithm 2.1** *(to generate a sliding twist)*

*Step 1. The pseudo-hyperbolic space rolls over the vector $u = (0, \varphi, 0)$. Control: $u(t) = u$.*

*Step 2. The pseudo-hyperbolic space rolls over a line segment parallel to the $z$-axis, with length $-\theta/\sinh(\varphi)$. Control: $u(t) = (0, 0, -1)$.*

*Step 3. The pseudo-hyperbolic space rolls back over a line segment parallel to the $y$-axis, with length $|\varphi|$. Control: $u(t) = (0, -\varphi, 0)$.*

*Step 4. The pseudo-hyperbolic space rolls on a circumference centred at $(0, \coth(\varphi), \frac{\theta}{\sinh(\varphi)})$, with radius equal to $|\coth(\varphi)|$, describing an angle $\theta$.*
*Control: $u(t) = \cosh(\varphi)(0, \sin(\sinh(\varphi)(t - 2 - \frac{\theta}{\sinh(\varphi)})), \cos(\cosh(\varphi)(t - 2 - \frac{\theta}{\sinh(\varphi)})))$.* □

Consider an arbitrary vector $s \in T_{p_0} H_0^2(1)$. We now present a rolling motion that generates a pure slip of $H_0^2(1)$ over $T_{p_0}^{\text{aff}} H_0^2(1)$, correspondent to the displacement $s$. Set $\varphi \in I\!R^-$ and denote by $T_\varphi(\theta)$ the translation operated on $H_0^2(1)$ when a sliding twist, with rotation angle $\theta$ and auxiliar value $\varphi$, is generated.

**Algorithm 2.2** *(to generate a pure slip)*

*Step 1. Perform the rolling motion along the vector $u = -T_\varphi(\pi) + \frac{1}{2}s$. Control: $u(t) = u$.*

*Step 2. Perform the rolling that generates the sliding twist of angle $\theta = \pi$, using the auxiliary value $\varphi$. (At the end the "midpoint" of the vector $s$ is reached.)*

*Step 3. Repeat the first step, that is, roll the pseudo-hyperbolic space again along the vector $u$, applied at the midpoint of $s$.*

*Step 4: Repeat the second step, to generate another sliding twist of angle $\pi$.*

□

The figure below illustrates the two previous algorithms.

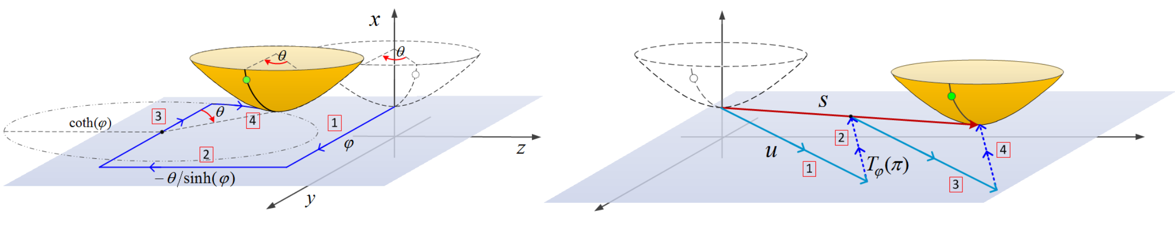


Figure 1: On the left a sliding twist is generated. On the right is generated a pure slip.

## References


[1] A. Marques, F. Silva Leite (2012), *Rolling a pseudohyperbolic space over the affine tangent space at a point.* Proc. CONTROLO'2012, paper 36, 16-18, Portugal.

[2] A. Marques, F. Silva Leite (2018), *Constructive Proof for Complete Controllability of a Rolling Pseudo-Hyperbolic Space.* Proc. CONTROLO'2018, paper 57, 19-24, Portugal.