

Geometric hydrodynamics

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In 1757 Leonhard Euler formulated the partial differential equations governing the motion of an incompressible inviscid fluid—the Euler equations—that every undergraduate in mechanics study today. In 1901, motivated by the dynamics of rotating rigid bodies in liquids, Henry Poincaré wrote down a system of ordinary differential equations describing mechanical systems evolving on finite-dimensional Lie groups. Then, in 1966, Vladimir Arnold made an extraordinary discovery connecting the work of Euler and Poincaré: the Euler equations can be interpreted as a Riemannian geodesic equation on the infinite-dimensional ‘Lie group’ of volume preserving diffeomorphisms. Arnold’s discovery gave rise to a new branch of mathematics today called *geometric hydrodynamics* (or sometimes *topological hydrodynamics*). It turned out that, not only the Euler equations, but many PDE in mathematical physics take the form of geodesic equations on diffeomorphism groups; examples include the KdV, Camassa–Holm, Landau–Lifschitz, and magnetohydrodynamic equations. The geometric insights provided by Arnold’s approach yield, for example, a better understanding of stability (by studying sectional curvature) and existence and uniqueness of solutions (by proving smoothness of the geodesic spray in the category of infinite-dimensional Banach manifolds).

The purpose of my lectures is to give a full account of Arnold’s beautiful discovery. If time permits I shall also discuss fruitful connections to other fields, such as optimal mass transport and quantum mechanics.

References

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- [2] L. Euler, Principia motus fluidorum, *Novi Commentarii academiae scientiarum Petropolitanae* **6** (1761), 271–311.
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