A Necessary Condition for Local Controllability

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Local controllability around an equilibrium is an important notion within control theory. Necessary or sufficient conditions for small-time local controllability (STLC) have been much investigated in the last decades. Some powerful sufficient conditions have been stated; however, most necessary conditions for STLC are more specific and deal with scalar-input control systems, including the classical result from Sussmann in [3].

The purpose of this talk is to extend this necessary condition to a particular class of systems with two controls, in which the field associated to the second control vanishes at the equilibrium point. In this case, the second control may allow better local controllability in some sense, provided the control vector fields verify another Lie bracket hypothesis.

Let f_0, f_1 be analytic vector fields on $\mathbf{R}^n$. Consider the control-affine system with scalar-input control

$$\dot{z} = f_0(z) + u_1(t) f_1(z), \tag{1}$$

with $f_0(0) = 0$ (meaning that $(0,0)$ is an equilibrium point for the system).

Definition 1 (STLC)**.** *The control system* (1) *is* STLC at $(0,0)$ *if, for every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that, for every* z_0, z_1 *in the ball centered at* 0 *with radius* η*, there exists a control* $u(\cdot)$ *in* $L^\infty([0,\varepsilon])$ *such that the solution of the control system* $z(\cdot) : [0,\varepsilon] \to \mathbf{R}^n$ *of* (1) *satisfies* $z(0) = z_0$*,* $z(\varepsilon) = z_1$*, and*

$$\|u\|_{L^\infty} \leqslant \varepsilon\,.$$

Note that this definition, used by Coron in [1], requires smallness both in time and in control. Nevertheless, another definition that only requires boundedness (and not smallness) of the control can be found in the works of Hermes and Sussmann [3] among others. This second definition, while not equivalent to the first one, is sometimes called STLC as well. In order to avoid the confusion, we will call it α-STLC:

Definition 2 (α-STLC)**.** *Let* $\alpha \geqslant 0$*. The control system* (1) *is* α-STLC at $(0,0)$ *if, for every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that, for every* z_0, z_1 *in the ball centered at* 0 *with radius* η*, there* *exists a control* $u(\cdot)$ *in* $L^\infty([0,\varepsilon])$ *such that the solution of the control system* $z(\cdot) : [0,\varepsilon] \to \mathbf{R}^n$ *of* (1) *satisfies* $z(0) = z_0$, $z(\varepsilon) = z_1$, *and*

$$\|u\|_{L^\infty} \leqslant \alpha + \varepsilon\,.$$

Let us call S_1 the subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ spanned by all the Lie brackets of f_0, f_1 containing f_1 at most one time, and $S_1(0)$ the subspace of $\mathbf{R}^n$ spanned by the value at 0 of the elements of S_1. The classical result from Sussmann states that the value at 0 of a certain bracket in which f_1 appears two times needs to belong to $S_1(0)$ for any STLC to hold.

Theorem 1. *Assume that* $[f_1, [f_0, f_1]](0) \notin S_1(0)$. *Then,* (1) *is not STLC(*α*) for any* α.

Our result proposes an extension of this result to the case of systems with a second scalar control u_2 against a vector field f_2 that vanishes at the equilibrium point. Consider the new system

$$\dot{z} = f_0(z) + u_1(t)f_1(z) + u_2(t)f_2(z) \tag{2}$$

with $f_0(0) = 0$ and $f_2(0) = 0$.

Similarly to the previous case, we call R_1 the subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ spanned by all the Lie brackets of f_0, f_1, f_2 containing f_1 at most one time, and $R_1(0)$ the subspace of $\mathbf{R}^n$ spanned by the value at 0 of the elements of S_1.

Theorem 2. *Assume that* $[f_1, [f_0, f_1]](0) \notin R_1(0)$. *Then,*

1. *if* $[f_1, [f_0, f_1]](0) \in \mathrm{Span}(R_1(0), [f_1, [f_2, f_1]](0))$, (2) *is not STLC.*
2. *else,* (2) *is not STLC(*α*) for any* α.

Remark 1. *Note how adding the second control* u_2 *changes the controllability level of the system in the first case. Provided that the right Lie brackets are linked at 0, the system is not STLC, but might be* α*-STLC for some* $\alpha > 0$*, which is never the case without a second control.*

The proof, which we will develop during the presentation, is based on the representation of trajectories through the Chen-Fliess series (see again [3]).

We will give an illustration of this result on the equations describing the 2D dynamics of a magnetic-driven elastic micro-swimmer made of two rigid links, studied in [2]. The equations of motions may be written as a control system with two control inputs (the two components of the magnetic field in the motion plane), that fits the hypothesis $f_0(0) = f_2(0) = 0$. The theorem ensures that the micro-swimmer is not STLC, but it is shown in [2] that it is α-STLC, with α depending on the physical parameters of the system.

References

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