On the topology of bracket-generating distributions

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A smooth manifold M can be endowed with many different geometric structures, many of which interact in interesting ways with the topology of the manifold itself. This is the case, for instance, of metrics with prescribed curvature, tight contact structures, or taut foliations.

Distributions are a type of geometric structure that arise naturally in Physics and, more particularly, in Control Theory. They are simply a (constant rank and smoothly varying) choice of linear subspace $\xi_p \subset T_pM$ at each $p \in M$. We can imagine that a particle in M is only allowed to move along the directions determined by the distribution ξ; as such, they are suitable for modelling systems with (linear) constraints.

Distributions, much like metrics, possess many pointwise differential invariants, many of which arise from the Lie bracket. Indeed, we can apply the Lie bracket to two vector fields tangent to a distribution ξ and see whether the resulting vector field is tangent as well. If this is always true, we say that ξ is *involutive*. The celebrated theorem by Frobenius says that ξ is involutive if and only if it defines a foliation. We can therefore use the Lie bracket to construct differential invariants that measure the non-involutivity of ξ. These are called *curvatures*.

As topologists, we are interested in the following question: "what is the homotopy type of the space of distributions with given curvatures?". That is, we fix (some of) the differential invariants and we try to see whether we can understand all distributions with such invariants.

A particular family of techniques that has proven to be incredibly fruitful in this context is the *h-principle* [3, 4]. I will introduce some of the key ideas behind the h-principle philosophy to prove M. Gromov's classical result on the classification of distributions in open manifolds. We will then turn our attention to close manifolds and go over Y. Eliashberg's result on contact structures in closed 3-manifolds [2]. If time allows I will explain some more recent developments in Contact Topology [5] and Engel Topology [1, 6].

This course will hopefully motivate the attendees to look a bit further into this beautiful topic. I will only be able to scratch the surface and therefore many interesting aspects I won't be able to discuss. In particular, I won't get into the rigidity aspects of Contact/Symplectic Topology (holomorphic curves, generating functions, or microlocal sheaves), nor into the more geometric viewpoint regarding distributions (studying local models *à la* Cartan, or looking at energy functionals and their extremals, to name a few). Many different and very active areas of mathematics converge here.

References

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