On the Controllability of Rolling Pseudo-Hyperbolic Space
A Constructive Proof

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Abstract

This poster presents some recent results [2] concerning a constructive proof for complete controllability of the rolling motion of a 2-dimensional pseudo-hyperbolic space, with index zero, over the affine space associated with the tangent space at a point. It is assumed that this rolling motion has the constraints of no-twist and no-slip (pure rolling). We describe a procedure that shows how to move the 2-dimensional pseudo-hyperbolic space up to any admissible configuration, by rolling without violating the constraints. This is accomplished by generating the forbidden motions of twist and slip using only pure rolling motions.

1 Establishing the problem

Consider the matrix $J_\kappa = \mathrm{diag}(-I_\kappa, I_{n-\kappa})$. The formula $\langle u, w\rangle_\kappa = u^\top J_\kappa w$ defines a scalar product on $\mathbb{R}^n$, making it a pseudo-Riemannian manifold, which we denote by $\mathbb{R}^n_\kappa$.

Associated with J_κ, one also defines the group $\mathrm{O}_\kappa(n) = \left\{R \in GL(n, \mathbb{R}) : R^\top J_\kappa R = J_\kappa\right\}$. Each isometry of $\mathbb{R}^n_\kappa$ has a unique expression as $x \mapsto Rx + s$, with $R \in \mathrm{O}_\kappa(n)$ and $s \in \mathbb{R}^n$. We will represent the isometries of $\mathbb{R}^n_\kappa$ by pairs (R, s).

The *pseudo-hyperbolic space* in $\mathbb{R}^{n+1}_{\kappa+1}$ is the hyper-quadric, with index κ, dimension n and radius r, defined by

$$H^n_\kappa(r) = \left\{p \in \mathbb{R}^{n+1}_{\kappa+1} : \langle p, p\rangle_{\kappa+1} = -r^2\right\}.$$

The *affine tangent space* to $H^n_\kappa(r)$ at a point p_0 is $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r) = \left\{p_0 + v : v \in T_{p_0} H^n_\kappa(r)\right\}$.

The main result about the rolling motion of $H^n_\kappa(r)$ over $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r)$ was proved in [1] and is presented next. Let p_0 be a point in $H^n_\kappa(r)$ and $t \in [0, \tau] \mapsto u(t) \in \mathbb{R}^{n+1}_{\kappa+1}$ a (piecewise) smooth function satisfying $\langle u(t), p_0\rangle_{\kappa+1} = 0$. If $R(t)$ and $s(t))$ form the solution-curve of

$$\begin{cases} \dot{s}(t) &= r^2 u(t) \\ \dot{R}(t) &= R(t)\left(-u(t)p_0^\top + p_0 u^\top(t)\right) J_{\kappa+1} \end{cases}, \qquad (*)$$

satisfying $(R(0), s(0)) = (I_{n+1}, 0)$, then $t \in [0, \tau] \mapsto X(t) = (R^{-1}(t), s(t))$ is a rolling map of $H^n_\kappa(r)$ over $T^{\mathrm{aff}}_{p_0} H^n_\kappa(r)$, without slipping or twisting.

The kinematic equations $(*)$ can be seen as a control system, where the controls are played by the components of the function u, and it was shown in [1] that this system is controllable. However, the proof of the controllability presented in [1] is not constructive, i.e., does not specify how to reach any admissible configuration. Thus, it makes sense to present a constructive proof of the controllability property, which is precisely the purpose of this work in the case where $H^n_\kappa(r) = H^2_0(1)$.

2 Moving $H^2_0(1)$ up to any admissible configuration $(\widetilde{R}, \widetilde{s})$

Consider $H^2_0(1) = \{(x, y, z) : x^2 - y^2 - z^2 = 1,\ x > 0\}$, and, without loss of generality, $p_0 = (1, 0, 0)$. The following algorithm is a procedure to move $H^2_0(1)$ up to any given configuration $(\widetilde{R}, \widetilde{s})$, when pure slips and pure twists are allowed. $\widetilde{R}$ belongs to the identity component of $\mathrm{O}_1(3)$ and $\widetilde{s} \in T_{p_0} H^2_0(1)$.

For $\varphi \in \mathbb{R}$, define

$$x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix}, \ y(\varphi) = \begin{bmatrix} \cosh(\varphi) & 0 & \sinh(\varphi) \\ 0 & 1 & 0 \\ \sinh(\varphi) & 0 & \cosh(\varphi) \end{bmatrix} \text{ and } z(\varphi) = \begin{bmatrix} \cosh(\varphi) & \sinh(\varphi) & 0 \\ \sinh(\varphi) & \cosh(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose that $\widetilde{R} = x(a_1) y(a_2) z(a_3)$. Then, the desired motion can be performed as explained next.

Algorithm 2.1. ***(to produce a motion up to the configuration*** $(\widetilde{R}, \widetilde{s})$***)***

Step 1 *Execute the rolling motion determined by the control* $u(t) = (0, -a_3, 0)$. *The pseudo-hyperbolic space rolls along this vector, to the configuration* $(z(a_3), (0, -a_3, 0))$.

Step 2 *Execute the rolling motion determined by the control* $u(t) = (0, 0, -a_2)$. *The pseudo-hyperbolic space rolls along this vector, to the configuration* $(y(a_2)z(a_3), (0, -a_3, -a_2))$.

Step 3 *Execute the pure twist correspondent to the rotation of angle* a_1 *around the* x*-axis. This moves the pseudo-hyperbolic space to the configuration* $(x(a_1)y(a_2)z(a_3), (0, -a_3, -a_2))$.

Step 4 *Execute a pure slip over the vector* $(0, a_3, a_2)$. *This moves the pseudo-hyperbolic space to the configuration* $(\widetilde{R}, 0)$.

Step 5 *Execute a pure slip over the vector* $\widetilde{s}$. *This moves the pseudo-hyperbolic space up to the desired final configuration* $(\widetilde{R}, \widetilde{s})$.

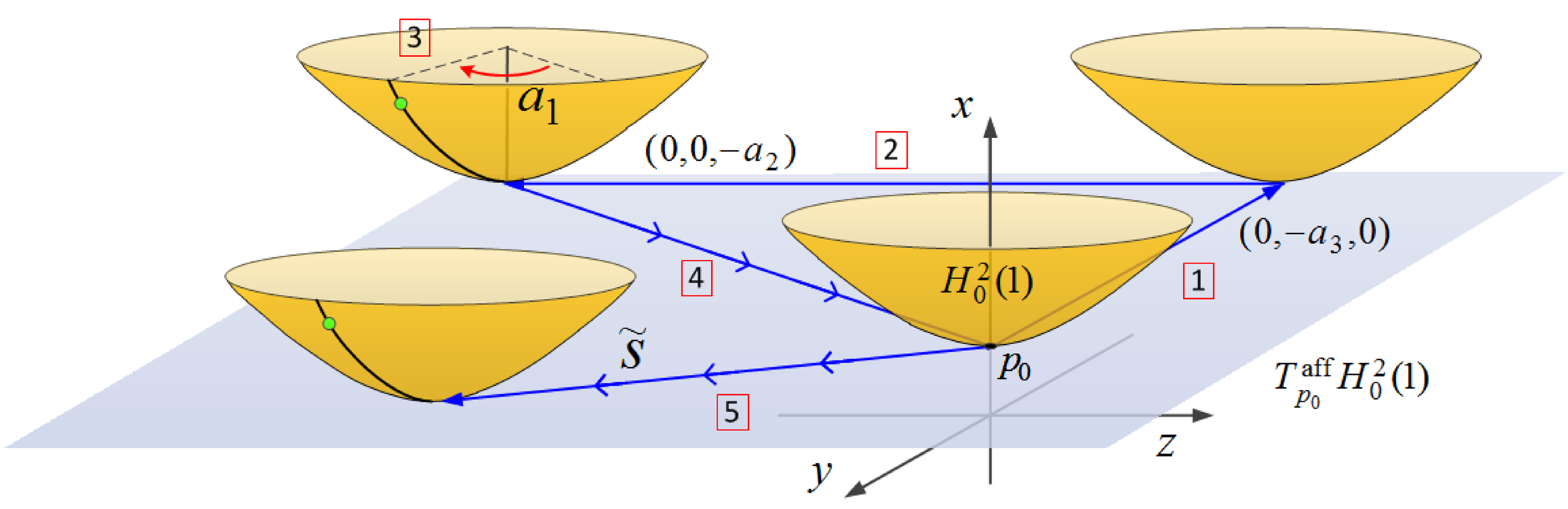

Figure 1: Moving the pseudo-hyperbolic space to the configuration $(x(a_1)y(a_2)z(a_3), \widetilde{s})$.

We notice that in general $\widetilde{R} = \prod_{i=1}^{m} x(a_1{}^i) y(a_2{}^i) z(a_3{}^i)$, where $a_1^i, a_2^i, a_3^i \in \mathbb{R}$ and $m \in \mathbb{N}$. But, when $m > 1$, the desired motion is achieved with successive iterations of the first four previous steps, before the final step 5 is performed.

3 Answers to two the essential questions

The previous algorithm reduces the constructive proof of controllability to showing how it is possible to replace the forbidden pure twists and pure slips by pure rolling motions, to obtain the same effect. Therefore, the key issues of this work can be formulated as follows:

- How to generate a pure twist or a sliding twist associated with any given angle θ?
- How to generate a pure slip associated with any given displacement $s \in T_{p_0} H^2_0(1)$?

3.1 Generating a sliding twist

Next we present a rolling motion that generates a sliding twist of $H^2_0(1)$ over $T^{\mathrm{aff}}_{p_0} H^2_0(1)$, corresponding to a rotation θ around the x-axis. For that, we start by choosing any non-zero auxiliary value φ, having opposite sign to the given angle θ.

Algorithm 3.1. ***(to generate a sliding twist)***

Step 1 *Perform the rolling motion given by* $u(t) = (0, \varphi, 0)$. *The pseudo-hyperbolic space rolls over a line segment parallel to the* y*-axis, with length* $|\varphi|$.

Step 2 *Perform the rolling motion given by* $u(t) = (0, 0, \theta/\sinh(\varphi))$. *The pseudo-hyperbolic space rolls over a line segment parallel to the* z*-axis, with length* $-\theta/\sinh(\varphi)$.

Step 3 *Perform the rolling motion given by* $u(t) = (0, -\varphi, 0)$. *The pseudo-hyperbolic space rolls back over a line segment parallel to the* y*-axis, with length* $|\varphi|$.

Step 4 *Perform the rolling motion over the circumference centred at* $(0, \coth(\varphi), \frac{\theta}{\sinh(\varphi)})$*, describing the angle* θ*, up to the final configuration:* $\left(x(\theta), (0, \coth(\varphi)(1 - \cos(\theta)), -\coth(\varphi)\sin(\theta) + \frac{\theta}{\sinh(\varphi)})\right)$.

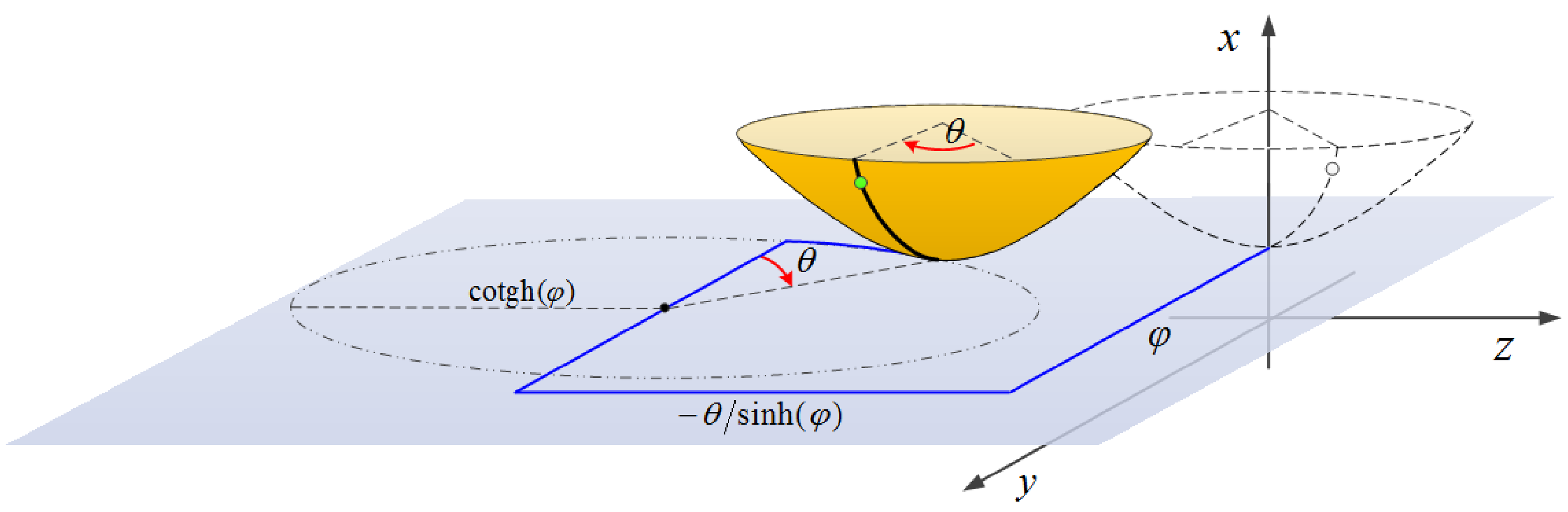

Figure 2: Generating a sliding twist with angle θ.

3.2 Generating a pure slip

We now present a rolling motion that generates a pure slip of $H^2_0(1)$ over $T^{\mathrm{aff}}_{p_0} H^2_0(1)$, corresponding to a given displacement $s \in T_{p_0} H^2_0(1)$. Set $\varphi \in \mathbb{R}^-$ and denote by $T_\varphi(\theta)$ the translation operated on $H^2_0(1)$ when a sliding twist, with rotation angle θ and auxiliar value φ, is generated.

Algorithm 3.2. ***(to generate a pure slip)***

Step 1 *Perform the rolling motion along the vector* $u = -T_\varphi(\pi) + \frac{1}{2}s$.

Step 2 *Perform the rolling that generates the sliding twist of angle* $\theta = \pi$*, using the auxiliary value* φ*. (At the end the "midpoint" of the vector* s *is reached.)*

Step 3 *Repeat the first step, that is, roll the pseudo-hyperbolic space again along the vector* u*, applied at the midpoint of* s.

Step 4 *Repeat the second step, to generate another sliding twist of angle* π*, up to the final configuration:* (I_3, s).

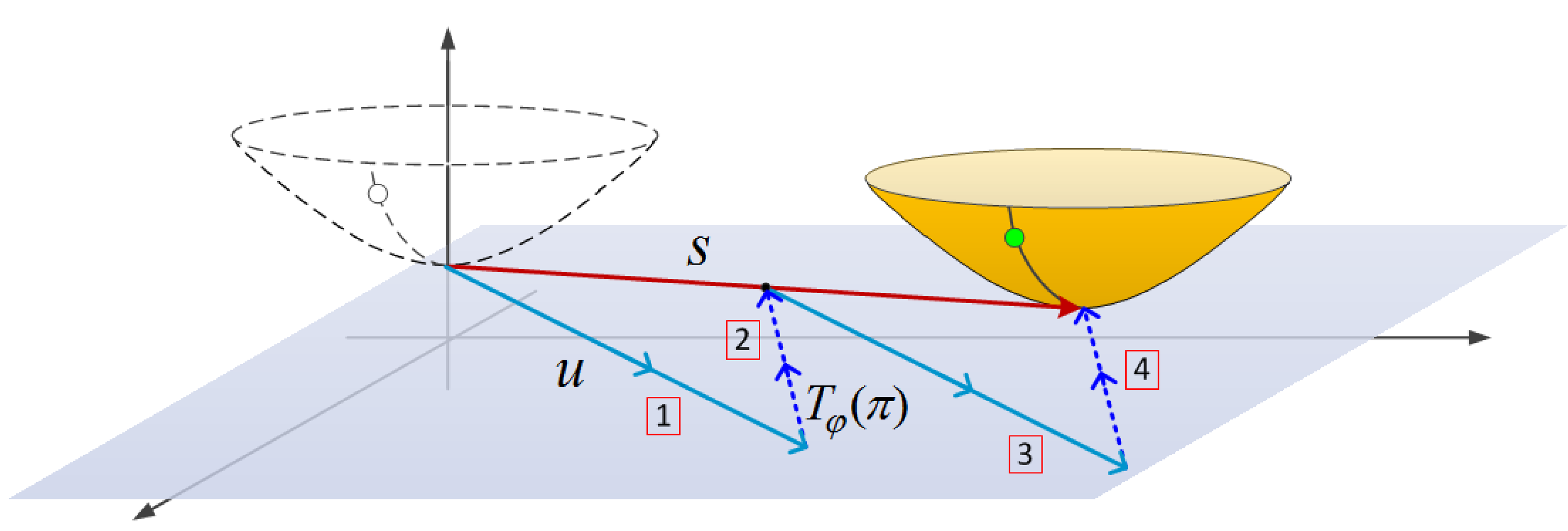

Figure 3: Generating a pure slip with displacement s.

Conclusions

- A constructive proof for complete controllability of the rolling system corresponds to show how pure twists or sliding twists and pure slips can be replaced by rolling without twisting or slipping.
- Sliding twists and pure slips can be generated by means of rolling motions on straight lines and circumferences, with the 4-step sequences described in algorithms 3.1 and 3.2.

References

[1] A. Marques and F. Silva Leite . Rolling a pseudohyperbolic space over the affine tangent space at a point. *Proc. CONTROLO'2012*, pages 123–128 (paper 36), 2012.

[2] A. Marques and F. Silva Leite . Constructive proof for complete controllability of a rolling pseudo-hyperbolic space. *Proc. CONTROLO'2018*, pages 19–24 (paper 57), 2018.