Exact discrete lagrangian for constrained mechanics: an open problem

Anahory Sena, Alexandre[1], Martín de Diego, David[1], Marrero, Juan Carlos[2]

[1]Instituto de Ciencias Matematicas, [2]Universidad de La Laguna

Introduction

A *nonholonomic constraint* on a mechanical system is a nonintegrable distribution $\mathcal{D}$ on Q and it is locally given by

$$\mu^a_i(q)\dot{q}^i = 0,\ 1 \leqslant a \leqslant k.$$

Nonholonomic dynamics is given by the *Lagrange-d'Alembert equations* for $L : TQ \to \mathbb{R}$, whose local expression is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \mu^a_i(q) \qquad (1)$$

$$\mu^a_i(q)\dot{q}^i = 0, \qquad (2)$$

for some Lagrange multipliers λ_a, to be determined. Under suitable regularity conditions, the solutions are integral curves of a well-defined second-order vector field $\Gamma_{nh} \in \mathfrak{X}(\mathcal{D})$.

Lagrangian discrete mechanics

Let $L_d : Q \times Q \to \mathbb{R}$ be the*discrete lagrangian function* and the *discrete path space* be

$$C_d(Q) = \{q_d = \{q_k\}_{k=0}^N \,|q_k \in Q\},$$

with q_0 and q_N fixed. The *discrete action map* is defined to be the map $S_d : C_d(Q) \to \mathbb{R}$,

$$S_d(q_d) = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}). \qquad (3)$$

The *discrete Hamilton's principle* implies the *discrete Euler-Lagrange equations* (see [4])

$$D_2L_d(q_{k-1}, q_k) + D_1L_d(q_k, q_{k+1}) = 0, \quad \text{for all } k = 1, \dots, N-1. \qquad (4)$$

By choosing the *exact discrete lagrangian* for a Lagrangian L

$$L^e_d(q_0, q_1, h) = \int_0^h L(q_{0,1}(t), \dot{q}_{0,1}(t))\, dt, \qquad (5)$$

where $q_{0,1}(t)$ is the solution of standard Euler-Lagrange equations with $q_{0,1}(0) = q_0$ and $q_{0,1}(h) = q_1$, the solution of (4) for L^e_d becomes exact in the sense that the discrete path is the exact sequence $\{q_{0,1}(kh), k = 0, 1, 2, \dots\}$. Moreover, the converse holds.

Nonholonomic exponential map

Let L be a kinetic lagrangian L with nonholonomic flow $\varphi^{\Gamma_{nh}}_t : D \to D$. The nonholonomic exponential map

$$\exp^{\Gamma_{nh}}_h : \mathcal{U} \subseteq \mathcal{D} \to Q \times Q$$
$$v_{q_0} \mapsto (q_0, \tau \circ \varphi^{\Gamma_{nh}}_h(v_{q_0}))$$

can be proven to be a smooth local embedding and therefore a local diffeomorphism into its image, denoted by $\mathcal{M}^h_d$. Its inverse is denoted by $R^{e^-}_h : \mathcal{M}^h_d \longrightarrow \mathcal{D}$.

EDLA equations

Take the *exact discrete Lagrangian* defined by (5), but now $q_{0,1}(t; q_0, q_1) = \tau \circ \varphi^{\Gamma_{nh}}_h(R^{e^-}_h(q_0, q_1))$ is a solution of (1) and (2). If $(q_0, q_1), (q_1, q_2) \in \mathcal{M}^h_d$ are the points $q_k = q_{0,1}(kh)$ then

$$D_1L^e_h(q_1, q_2) + D_2L^e_h(q_0, q_1) + f^1_d(q_0, q_1) + f^0_d(q_1, q_2) = 0, \qquad (6)$$

is satisfied by the exact solution, where

$$\langle f^i_d(q_0, q_1), v_{q_i}\rangle = \int_0^h \langle \lambda_a(t)\mu^a, \frac{\partial q_{0,1}}{\partial q_i}(t; q_0, q_1) \cdot v_{q_i}\rangle\, dt. \qquad (7)$$

Here, $\frac{\partial q_{0,1}}{\partial q_1}(t; q_0, q_1) \cdot v_{q_1} := \frac{d}{ds}|_{s=0} q_{0,1}(t; q_0, q_1(s))$ with $v_{q_1} = q_1'(0)$ and a similar notation for the other expression. Note that we take $(q_0, q_1(s)) \in \mathcal{M}^h_d$ which restricts v_{q_1} to lie in a particular subspace of $T_{q_1}Q$. However, in [3] the authors argue that (6) is not a suitable integrator: it does not define a unique q_2 from (q_0, q_1).

Mechanics on general algebroids

A *skew-symmetric algebroid* is a a vector bundle $\tau_D : D \to Q$ together with a vector bundle map $\rho : D \to TQ$ called the *anchor* map, for which there is a bracket operation $[X, Y]$ defined on sections of D, i.e., $X, Y \in \Gamma(D)$. This structure generalizes several geometric constructions such as tangent bundles, Lie algebras, Atitah bundles or nonholonomic mechanical systems.

If we write

$$[X_a, X_b]_D = C^c_{ab}X_c \qquad \rho(X_a) = \rho^i_a \frac{\partial}{\partial q^i}$$

on local coordinates (q^i) on Q and coordinates (q^i, y^a) on D adapted to the submersion τ_D and associated to a local basis of sections $\{X_a\}$ then

$$\frac{dq^i}{dt} = \rho^i_a y^a \qquad (8)$$

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial y^b}\right) - \frac{\partial \ell}{\partial y^c} y^a C^c_{ab} - \rho^i_b \frac{\partial \ell}{\partial q^i} = 0, \qquad (9)$$

(9) are the *Euler-Lagrange* equations on the algebroid D for a lagrangian $\ell : D \to \mathbb{R}$, whereas (8) is the *admissibility condition* for the solution curve. In the particular case, where the algebroid is a tangent bundle or a Lie algebra, equations (9) are the standard Euler-Lagrange or Euler-Poincaré equations, respectively.

We have a "variational-type" principle to obtain (9) from the functional

$$\mathrm{J}_D(\gamma) = \int_0^h \ell(\gamma(t))\, dt, \qquad (10)$$

on admissible paths in D, i.e., $\rho \circ \gamma = \frac{d}{dt}(\tau_D \circ \gamma)$. Given a curve γ in D, take the complete lift of a section X of D over $\underline{\gamma} := \tau_D \circ \gamma$, i.e., $\tau_D \circ X = \underline{\gamma}$, locally defined by

$$X^c_\gamma(t) = X^a(t)\rho^i_a(\underline{\gamma}(t)) \left.\frac{\partial}{\partial q^i}\right|_{\gamma(t)} + \left[\dot{X}^a(t) + C^a_{bd}(\underline{\gamma}(t))\gamma^b(t)X^d(t)\right] \left.\frac{\partial}{\partial y^a}\right|_{\gamma(t)}. \qquad (11)$$

Then (see [5]) an admissible curve γ is a solution of (9) if and only if for every section X of D over $\underline{\gamma}$ and vanishing at the end-points we have

$$\langle d\mathrm{J}_D(\gamma), X^c_\gamma\rangle = 0.$$

An exact integrator

Given a projector $P : TQ \to \mathcal{D}$ (e.g., an orthogonal projector associated to the Riemannian metric defining the kinetic Lagrangian). Denote by $i : \mathcal{D} \hookrightarrow TQ$ the inclusion map.

The maps

$$\rho(X) = i(X), \quad [X, Y]_D = P([\rho(X), \rho(Y)]), \qquad (12)$$

where $X, Y \in \Gamma(D)$ and the bracket on the right-hand side is the Lie bracket of vector fields, define an algebroid structure on $\mathcal{D}$ (cf. [2]).

Define the exact discrete lagrangian $L^e_h : \mathcal{M}^h_d \longrightarrow \mathbb{R}$, by

$$L^e_h(q_0, q_1) = \int_0^h \ell(q_{0,1}(t), \dot{q}_{0,1}(t))\, dt = \int_0^h \ell(\gamma_{0,1}(t))\, dt,$$

where $\gamma_{0,1} := (q_{0,1}(t), \dot{q}_{0,1}(t)) \in \mathcal{D}$ is the unique nonholonomic solution with $q_{0,1}(0) = q_0$ and $q_{0,1}(h) = q_1$.

Define the discrete operators $\mathbb{B}L^e_h : \mathcal{M}^h_d \to \mathcal{D}^*$ and $\mathbb{F}L^e_h : \mathcal{M}^h_d \to \mathcal{D}^*$ called the *backward* and *forward discrete transforms*, respectively, defined by

$$\langle \mathbb{B}L^e_h(q_0, q_1), X_{q_0}\rangle = \langle d\mathrm{J}_D(\gamma_{0,1}), X^c_{\gamma_{0,1}}\rangle \qquad (13)$$

$$\langle \mathbb{F}L^e_h(q_0, q_1), Y_{q_1}\rangle = \langle d\mathrm{J}_D(\gamma_{0,1}), Y^c_{\gamma_{0,1}}\rangle \qquad (14)$$

where in (13) $X_{q_0} \in \mathcal{D}_{q_0}$, and $X^c_{\gamma_{0,1}}$ is the complete lift of any section X of $\mathcal{D}$ over $\underline{\gamma}$ satisfying $X(0) = X_{q_0}$ and $X(h) = 0$. Analogously, in (14), $Y_{q_1} \in \mathcal{D}_{q_1}$ and we pick any Y over $\underline{\gamma}$ satisfying $Y(0) = 0$ and $Y(h) = Y_{q_1}$.

Then, under suitable regularity conditions, the equation

$$\mathbb{F}L^e_h(q_0, q_1) + \mathbb{B}L^e_h(q_1, q_2) = 0. \qquad (15)$$

is an exact integrator for nonholonomic mechanics. However, (15) is not given by a discrete "variational-type" princple (such as in the standard case). Our goal is:

- construct a discrete variational principle for nonholonomic mechanics for which we can find an exact discrete lagrangian with properties similar to those of (5). We hope to do it in a framework similar to that in [1], by relaxing the condition of Lie groupoid, taking into account the particular structure of nonholonomic mechanics.
- compare with other integrators and formulate a "variational" error analysis.

References

[1] J.C. Marrero, D. Martín de Diego, E. Martínez,
On the exact discrete lagrangian function for variational integrators: theory and applications.
arXiv:1608.01586v1 [math.DG], (2016).

[2] M. Barbero-Liñán, M. de León, D. Martín de Diego, J.C. Marrero, M.C. Muñoz-Lecanda,
Kinematic reduction and the Hamilton-Jacobi equation,
J. Geom. Mech., **4** (2012) 207–237.

[3] R. McLachlan, M. Perlmutter,
Integrators for Nonholonomic Mechanical Systems,
J. Nonlinear Sci., **16** (2006) 283–328.

[4] J.E. Marsden, M. West,
Discrete mechanics and variational integrators,
Acta Numerica, **10** (2001), 1–159.

[5] K. Grabowska, J. Grabowski,
Variational calculus with constraints on general algebroids,
J. Phys. A: Mathematical and Theoretical, **41**(17) (2008), 175–204.

[6] K. Modin, O. Verdier,
How do nonholonomic integrators work?
arXiv:1707.04514v1 [math.NA], (2017).