The reveal of minimal ruled surfaces using control theory

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ABSTRACT: We seek minimal surfaces among the family of ruled surfaces in the Euclidean space [1]. Such surfaces are generated by a straight line, called generatrix, that moves along a curve, the directrix. We approach this problem using the techniques from control theory. In the literature the minimal revolution surface is often obtained as the solution to an optimal control problem. Nothing similar has been published for other families of minimal surfaces because partial differential equations cannot be avoided. This is why we use the k–symplectic formalism to recover the plane and the helicoid as extremals of an optimal control problem.

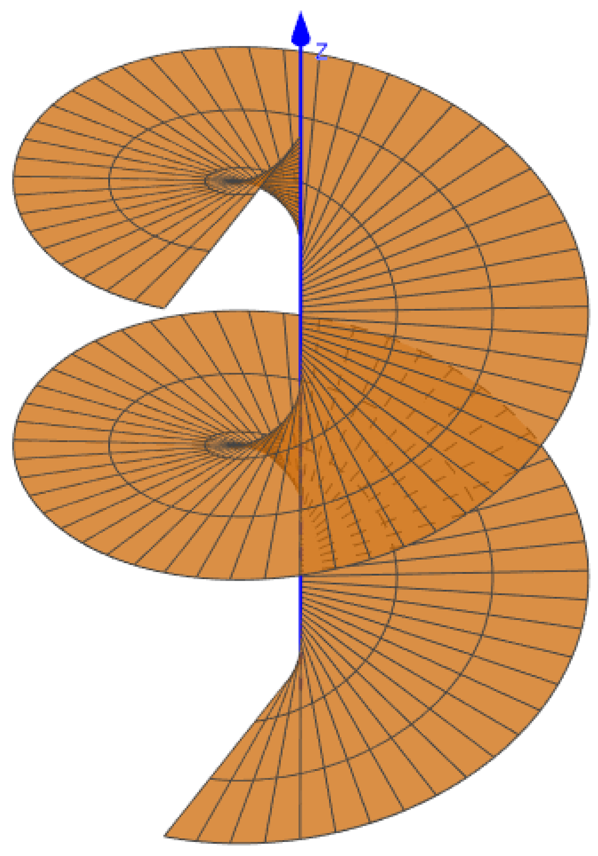
1 Introduction

Definition 1 A **ruled surface** is generated by straight lines and it can be described parametrically as

$$\begin{aligned}\sigma\colon D\subseteq\mathbb{R}^2 &\longrightarrow \mathbb{R}^3\\ (u,v)&\longmapsto \sigma(u,v)=\gamma(u)+v\,w(u)\end{aligned}$$

where $\gamma(u), w(u)$ are curves in $\mathbb{R}^3$.

Example 1 The **helicoid** $\sigma(u,v)=(0,0,bu)+v(a\cos u, a\sin u, 0)$, $a,b\in\mathbb{R}$, $D=\mathbb{R}\times[0,1]$.
For $b=0$, it becomes the plane.
In cylindrical coordinates, $\sigma(r,\theta)=(r,\theta,c\theta)$, where $c\in\mathbb{R}$, $D=[0,1]\times\mathbb{R}$.



Definition 2 A **minimal surface** in $\mathbb{R}^3$ is a surface with zero mean curvature.

Meusnier (1776): The helicoid and the plane are the only minimal ruled surfaces.

A surface is minimal if and only if it is a critical point of the area functional (Lagrange, 1760). The area of an explicit surface $S=\{(x,y,z(x,y))\,|\,(x,y)\in\mathrm{Dom}(z)\}$ is

$$A=\iint_{\mathrm{Dom}(z)}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy\,. \qquad (1)$$

2 Background in k–symplectic formalism

The partial derivatives in (1) make impossible to use classical optimal control theory, except for the minimal revolution surface (the catenoid).

A crash course on k–symplectic formalism [2]:
The k**-tangent bundle** of Q, T^1_kQ, is the Whitney sum:

$$T^1_kQ=TQ\oplus\overset{k}{\cdots}\oplus TQ.$$

The elements of T^1_kQ are k-tuples $(v_{1_q},\dots,v_{k_q})$ of vectors in T_qQ, $q\in Q$.
The **canonical projection** $\tau^{\mathbf{k}}_{\mathbf{Q}}\colon\mathbf{T^1_kQ}\to\mathbf{Q}$: $\tau^k_Q(v_{1_q},\dots,v_{k_q})=q$.
Local coordinates for T^1_kQ: (q^i,v^i_A), $A=1,\dots,k$, $i=1,\dots,\dim Q$.
A k**-vector field** on Q is a section $\mathbf{X}\colon Q\to T^1_kQ$ of τ^k_Q.
The **canonical projection** $\tau^{\mathbf{k;A}}_{\mathbf{Q}}(\mathbf{v_{1_q}},\dots,\mathbf{v_{k_q}})=\mathbf{v_{A_q}}$ associates $\mathbf{X}$ with

a family of vector fields on Q : $X_A=\tau^{k;A}_Q\circ\mathbf{X}\quad A=1,\dots,k.$

An **integral section of** $\mathbf{X}$ is a map $\sigma\colon\mathbb{R}^k\to Q$, $\mathbf{t}\to\sigma(\mathbf{t})$ s.t.

$$\mathrm{T}^1_k\sigma=\left(\frac{\partial\sigma}{\partial t^1},\dots,\frac{\partial\sigma}{\partial t^k}\right)_{\sigma(\mathbf{t})}=\mathbf{X}\circ\sigma\quad\text{where } \mathbf{t}=(t^1,\dots,t^k).$$

k-symplectic optimal control problem

$$\begin{array}{ccccc} & & & & T^1_kQ\\ & \nearrow^{(\mathrm{T}^1_k)(\pi_1\circ\phi)} & & \nearrow^{\mathbf{X}} & \downarrow{\scriptstyle\tau^k_Q}\\ \mathbf{I}=I_1\times\cdots\times I_k\subseteq\mathbb{R}^k & \xrightarrow{\phi} & Q\times U & \xrightarrow{\pi_1} & Q\end{array}$$

Let $L\colon Q\times U\to\mathbb{R}$ be the **cost function** and $u:\mathbf{I}\subset\mathbb{R}^l\to U$ are the **control functions**.
Find a map $\phi=(\sigma,u)\colon\mathbf{I}\to Q\times U$ passing through q_0 and q_f in Q s. t.

1. it is an integral section of $\mathbf{X}=(X_1,\dots,X_k)$, i.e.

$$\mathrm{T}^1_k(\pi_1\circ\phi)=\mathbf{X}\circ\phi,\quad\text{i.e.}\quad\frac{\partial\sigma^i}{\partial t^A}(\mathbf{t})=X^i_A(\phi(\mathbf{t}))=X^i_A(\sigma(\mathbf{t}),u(\mathbf{t})),$$

2. it minimizes the functional $\int_{I_1\times\cdots\times I_k}L(\tilde\phi(\mathbf{t}))\mathrm{d}t^1\wedge\cdots\wedge\mathrm{d}t^k$
among all the integral sections $\tilde\phi$ of $\mathbf{X}$ on $Q\times U$ passing through q_0 and q_f.

2.1 Hamilton-De Donder-Weyl equation

Let us denote by T^*Q the cotangent bundle of Q, and $(T^1_k)^*Q$ the k–cotangent bundle of Q.
Pontryagin's Hamiltonian function is a map $\mathbf{H}:(T^1_k)^*Q\times U\longrightarrow\mathbb{R}$ given by

$$\mathbf{H}(\mathbf{p},u)=\sum_{A=1}^k H_A(\mathbf{p},u),\qquad\text{where}\qquad H_A(\mathbf{p},u)=-L(q,u)+\sum_{j=1}^n p^A_jX^j_A(q,u).$$

Definition 3 A k–vector field $\mathbf{X}^*$ is said to be Hamiltonian if, for every control u, it satisfies Hamilton-De Donder-Weyl's equation,

$$\sum_{A=1}^k i_{X^{*\{u\}}_A}\omega^A=d\mathbf{H}. \qquad (2)$$

By expressing the Hamiltonian k–vector field in components,

$$X^*_A=(Y_A)^i\frac{\partial}{\partial q^i}+(Y_A)^C_j\frac{\partial}{\partial p^C_j},$$

equation (2) leads to the adjoint equations,

$$\sum_{A=1}^k(Y_A)^A_i=\sum_{A=1}^k\left(\frac{\partial L}{\partial q^i}-p^A_j\frac{\partial X^j_A}{\partial q^i}\right)\qquad\forall\,1\le i\le n. \qquad (3)$$

Pontryagin's maximum principle (PMP), as stated in [3], claims that the solution to the extended k–symplectic optimal control problem, that is, the optimal integral section, must be the integral section of a Hamiltonian k–vector field. In addition each Hamiltonian function H_A attains its supremum over the controls along the optimal integral section.

3 Contribution

In cylindrical coordinates the area integral in (1) becomes:

$$A=\iint_{\mathrm{Dom}(z)}L(r,\theta)rdrd\theta=\iint_{\mathrm{Dom}(z)}\sqrt{r^2(1+u^2)+v^2}drd\theta. \qquad (4)$$

We take as controls

$$u=\frac{\partial z}{\partial r},\qquad v=\frac{\partial z}{\partial\theta},$$

(r,θ) corresponds with the variables (t^1,t^2) in 2–symplectic formalism.
Without loss of generality, to minimize the integral (4) is equivalent to minimize the double integral of the non-autonomous integrand

$$L(r,u,v)=r^2(1+u^2)+v^2\,,$$

which depends explicitly on $t^1=r$.
Trick: Introduce a new variable $z_0=r$ such that

$$\frac{\partial z_0}{\partial r}=1\quad\text{and}\quad\frac{\partial z_0}{\partial\theta}=0$$

and the Lagrangian is autonomous.
Setting: $Q=(0,\infty)\times\mathbb{R}$ with local coordinates (z_0,z). $L(z_0,z,u,v)=z_0^2(1+u^2)+v^2\,.$
Pontryagin's Hamiltonian function:
$\mathbf{H}(z_0,z,p^1_0,p^1,p^2_0,p^2,u,v)=-L(z_0,z,u,v)+p^1_0\cdot1+p^1\cdot u+p^2_0\cdot0+p^2\cdot v=-2z_0^2(1+u^2)-2v^2+p^1_0+p^1u+p^2v.$
Dynamics: $X_1(z_0,z,u,v)=1\,\dfrac{\partial}{\partial z_0}+u\,\dfrac{\partial}{\partial z}$ and $X_2(z_0,z,u,v)=v\dfrac{\partial}{\partial z}\,.$
A weaker necessary condition, but not sufficient, to optimize Pontryagin's Hamiltonian over the control set as required by PMP, is

$$\frac{\partial\mathbf{H}}{\partial u}=0\Leftrightarrow-4z_0^2u+p^1=0\quad\text{and}\quad\frac{\partial\mathbf{H}}{\partial v}=0\Leftrightarrow-4v+p^2=0\,. \qquad (5)$$

For our particular problem, the adjoint equation corresponding to $i=2$ is

$$\frac{\partial p^1}{\partial r}+\frac{\partial p^2}{\partial\theta}=0\,. \qquad (6)$$

By imposing tangency conditions to (5), we get

$$\left\{\begin{aligned}\frac{\partial}{\partial r}(-4z_0^2u+p^1)&=0\\ \frac{\partial}{\partial\theta}(-4v+p^2)&=0\end{aligned}\right\}\Leftrightarrow\left\{\begin{aligned}-8z_0u-4z_0^2\frac{\partial u}{\partial r}+\frac{\partial p^1}{\partial r}&=0\\ -4\frac{\partial v}{\partial\theta}+\frac{\partial p^2}{\partial\theta}&=0\end{aligned}\right\}$$

Adding the right-hand side equations and using (6), we get:

$$-8z_0u-4z_0^2\frac{\partial u}{\partial r}-4\frac{\partial v}{\partial\theta}=0.$$

In order to solve the PDEs we first try with locally constant controls that simplify the above equation to

$$-8z_0u=0.$$

The possible solutions are either $r=z_0=0$ or $u=0$. The first possibility does not define a surface because one parameter must be identically zero. The second possibility implies that $z_r=0$. Since $v=z_\theta$, we have $z(r,\theta)=v(\theta-\theta_0)+z_0$. Note that if v **is different from zero, we obtain the parametrization of the helicoid. If** v **is identically zero, we obtain a plane, the degenerate minimal ruled surface.**

References

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