Dynamics of pendula hanging from a string

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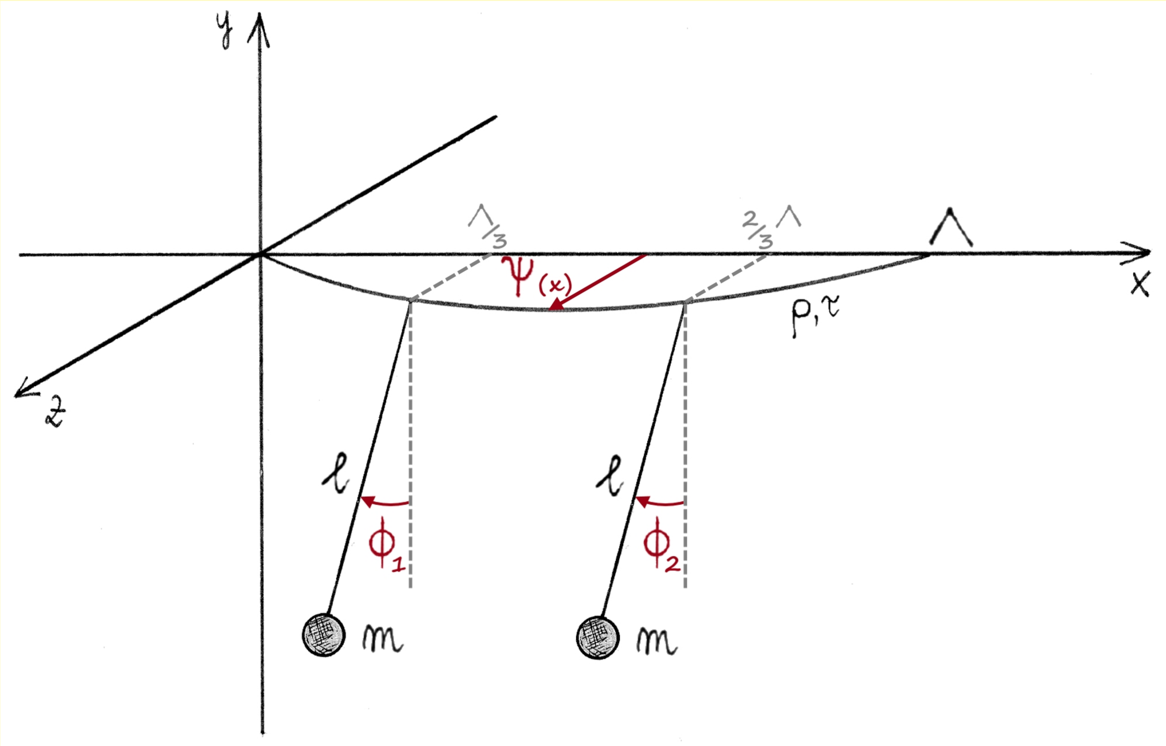
The Problem under Investigation



We consider *small oscillations* about the equilibrium

Q's : • **What is the dynamics?**

• **Will the pendula ever synchronize? Why?**

Lagrangian

$$L(\phi_1,\phi_2,\psi,\dot\phi_1,\dot\phi_2,\psi_t) = T(\dot\phi_1,\dot\phi_2,\psi_t) - V(\phi_1,\phi_2,\psi)$$

Kinetic energy:

$$T=\int_0^\Lambda\left\{\sum_{k=1}^{2}\frac{m}{2}\left[\ell^2\dot\phi_k^2+\psi_t^2+2\ell\dot\phi_k\psi_t\right]\delta(x-\tfrac{k\Lambda}{3})+\frac{\rho}{2}\psi_t^2\right\}dx$$

Potential energy:

$$V=\int_0^\Lambda\left\{\sum_{k=1}^{2}\left[\frac{1}{2}mg\ell\phi_k^2\right]\delta(x-\tfrac{k\Lambda}{3})+\frac{\tau}{2}\psi_x^2\right\}dx$$

Damping

Internal damping within the string [2]
$\rightarrow$ *viscoelastic* friction:

$$-\gamma\,\psi_{txx}(x,t)$$

Damping rates depend strongly on the frequency.

Damped normal modes

Look for solutions of the equations of motion of the form:

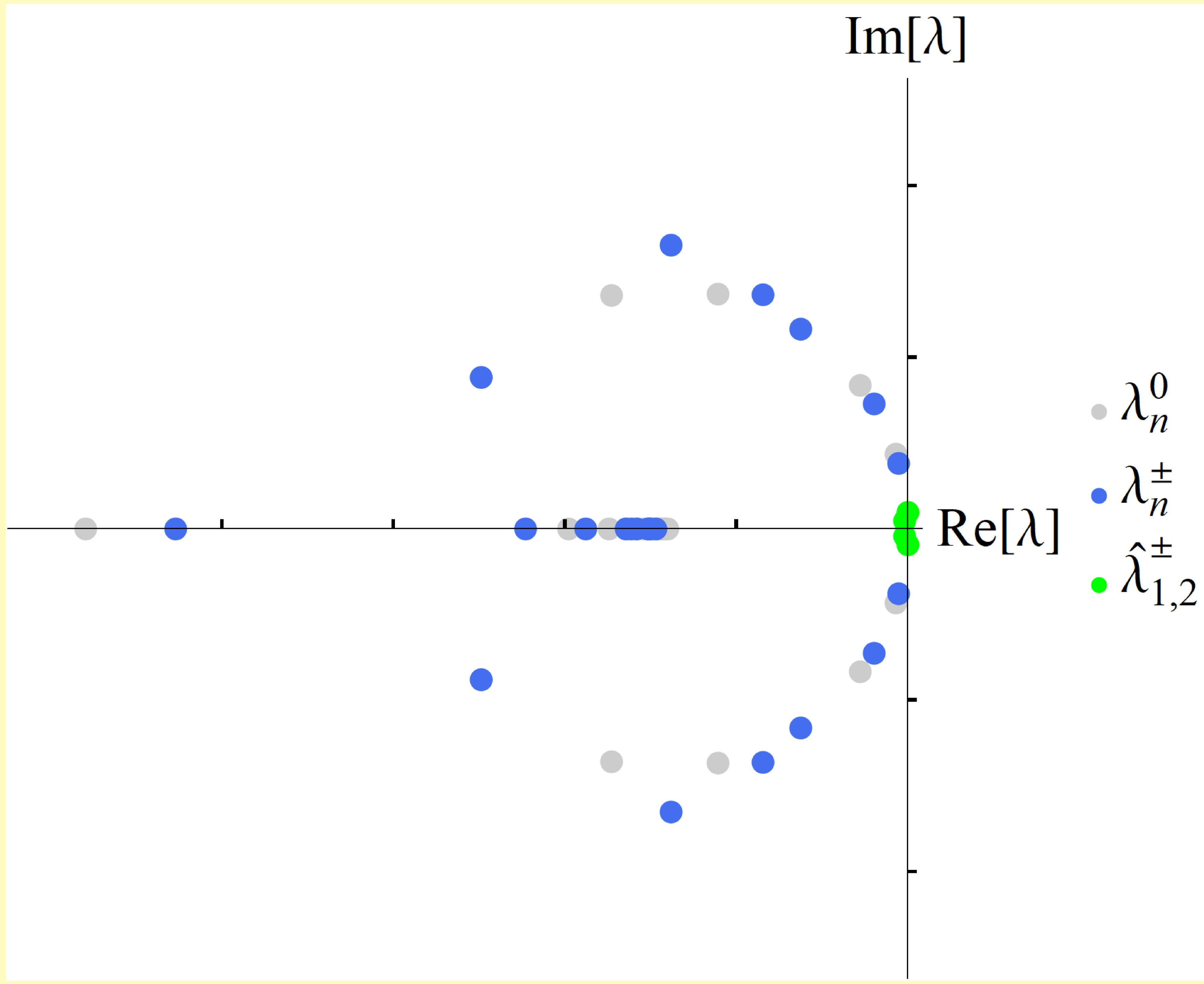
$$\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\\ \psi(x,t)\end{pmatrix} = \mathrm{Re}\left[\widehat\Phi_\lambda(x)e^{\lambda t}\right]$$

with $\lambda\in\mathbb{C}$, $\mathrm{Re}[\lambda]<0$ and $\widehat\Phi_\lambda:[0,\Lambda]\to\mathbb{C}^3$.

Results

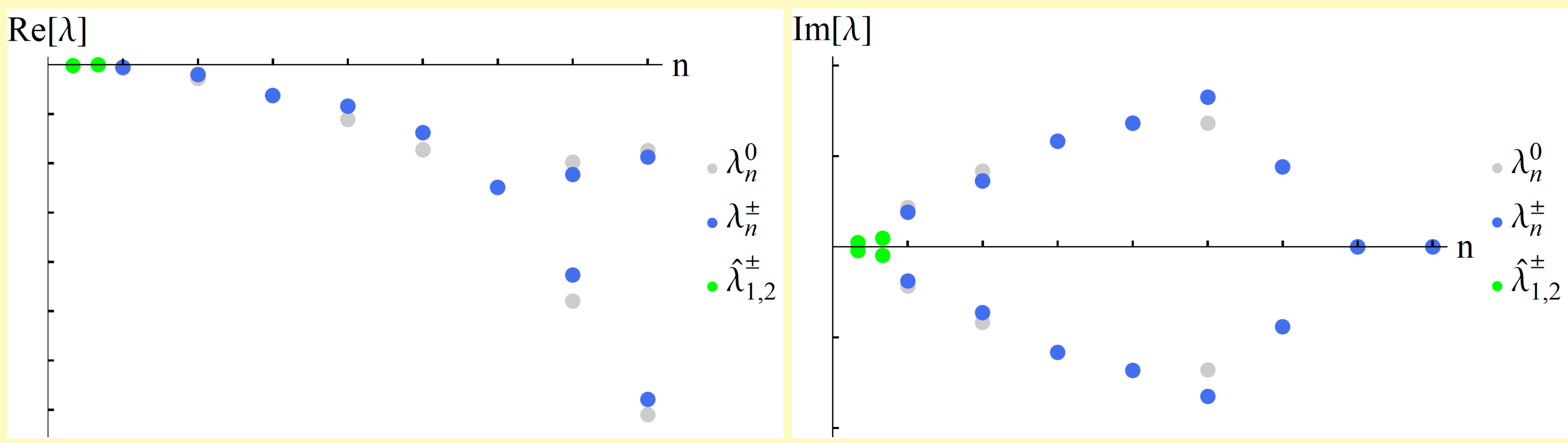
Spectrum

Proposition. *The spectrum consists of two countable families of eigenvalues* $\{\lambda_{3j}^\pm\}_{j\in\mathbb{N}_+}$, $\{\lambda_{3j-2}^\pm,\lambda_{3j-1}^\pm\}_{j\in\mathbb{N}_+}$ *and two pairs of special eigenvalues* $\widehat\lambda_1^\pm$, $\widehat\lambda_2^\pm$.



- $\lambda_{3j}^\pm$ coincide with the corresponding eigenvalues λ_{3j}^0 of the damped vibrating string. For these values the two pendula are at rest;
- $\lambda_{3j-k}^\pm$ $(k=1,2)$ are perturbations of the corresponding eigenvalues λ_{3j-k}^0 of the damped vibrating string and tend to them as $m\to 0$;
- $\widehat\lambda_{1,2}^\pm$ are perturbations of the pendulums' eigenfrequencies and tend to them as $\rho\to\infty$.

Real and imaginary parts of the dispersion relation:



- All eigenvalues have negative real parts and $|\mathrm{Re}[\lambda_n^\pm]|\gg|\mathrm{Re}[\widehat\lambda_{1,2}^\pm]|$ for every $n\geq 1$.
- Only a finite number of eigenvalues has a non-vanishing imaginary part.

Damped small oscillations

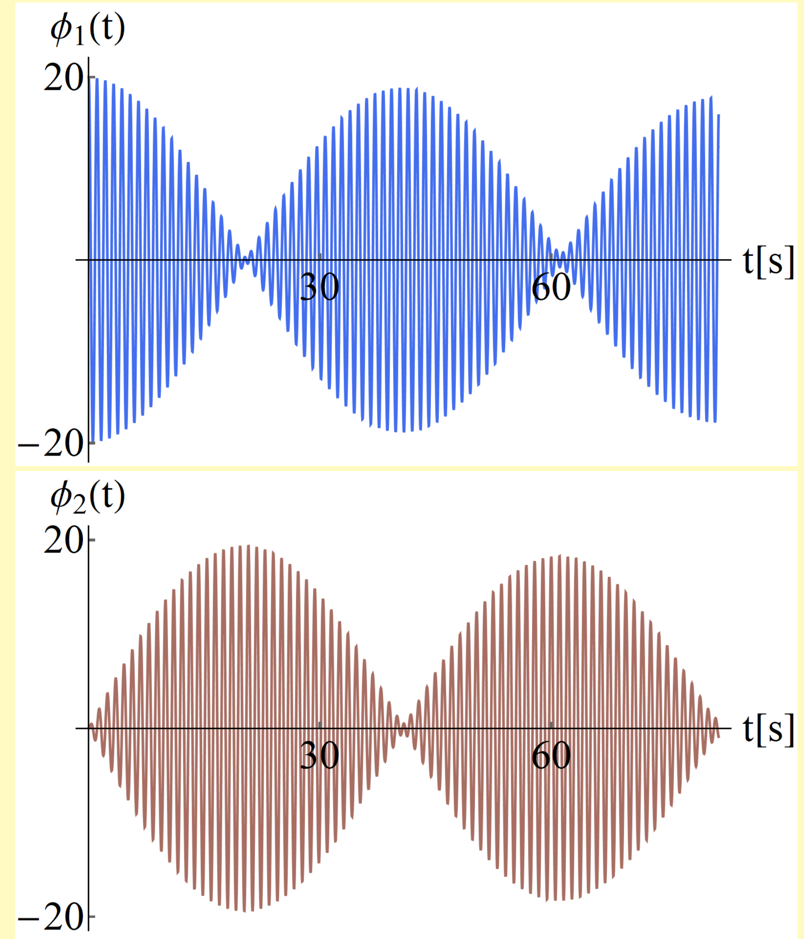
$$\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\\ \psi(x,t)\end{pmatrix} = \sum_{k=1}^{2}\mathrm{Re}\left[\widehat\Phi_{\widehat\lambda_k^+}(x)e^{\widehat\lambda_k^+ t}+\widehat\Phi_{\widehat\lambda_k^-}(x)e^{\widehat\lambda_k^- t}\right]+\sum_{n=1}^{+\infty}\mathrm{Re}\left[\widehat\Phi_{\lambda_n^+}(x)e^{\lambda_n^+ t}+\widehat\Phi_{\lambda_n^-}(x)e^{\lambda_n^- t}\right]$$

Beats and Synchronization

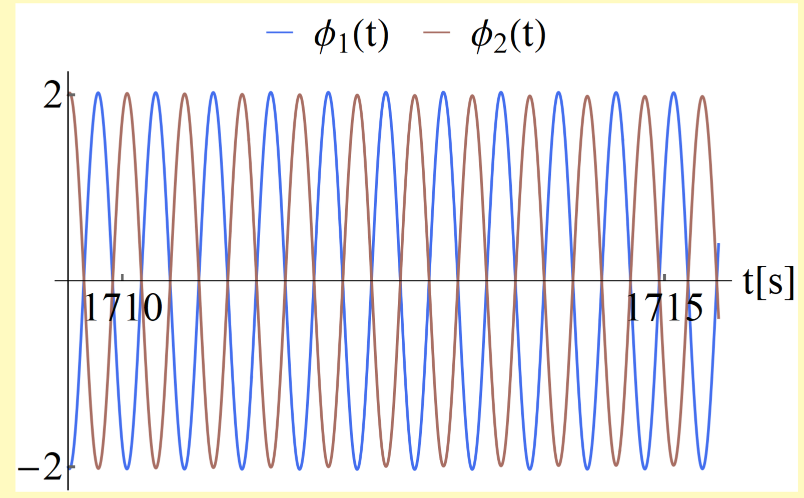
For long t:

$$\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\\ \psi(x,t)\end{pmatrix} \sim \sum_{k=1}^{2}\mathrm{Re}\left[\widehat\Phi_{\widehat\lambda_k^+}(x)e^{\widehat\lambda_k^+ t}+\widehat\Phi_{\widehat\lambda_k^-}(x)e^{\widehat\lambda_k^- t}\right]$$

High-frequency oscillations are damped out after a short transient. Thereafter, only a combination of a few (likely two) damped normal modes contributes and beating phenomena are observable.



For sufficiently large times, anti-phase synchronization between the two pendula occurs.



References

[1] S. Galasso. *Dynamics of pendula hanging from a string*. Master's Thesis, Università degli Studi di Padova (2018).

[2] D. L. Russell. *On Mathematical Models for the Elastic Beam with Frequency-Proportional Damping*. SIAM (1992), 125–169.

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