Geometric description of quantum states: Markovian dynamics and control of open quantum systems

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The Heisenberg picture of Quantum Mechanics

In its modern formulation, the Heisenberg picture of Quantum Mechanics presents observables of quantum systems as elements in a **Lie-Jordan algebra** $\mathcal{O}$. There exist two inner operations: a Lie bracket $[\![A,B]\!]$, and a Jordan product $A \odot B$.

States of quantum systems are normalised, positive linear functionals on $\mathcal{O}$:

$$\mathcal{S} = \{\rho \in \mathcal{O}^* \mid \rho(I) = 1;\ \rho(A^2) \geq 0, \forall A \in \mathcal{O}\} \subset \mathcal{O}^*.$$

For finite-dimensional systems, the dual space $\mathcal{O}^*$ presents a natural geometric structure. Observables are represented on $\mathcal{O}^*$ by real linear functions $f_A(\xi) = \xi(A)$. There exist **two contravariant** $(2,0)$**-tensor fields** Λ and R, respectively Poisson and symmetric, defined as

$$\Lambda(\mathrm{d}f_A, \mathrm{d}f_B)(\xi) = \xi([\![A,B]\!]) = f_{[\![A,B]\!]}(\xi),$$
$$R(\mathrm{d}f_A, \mathrm{d}f_B)(\xi) = \xi(A \odot B) = f_{A\odot B}(\xi),$$

Tensor fields Λ and R define respectively a Poisson bracket and a symmetric product of functions:

$$\{f,g\} = \Lambda(\mathrm{d}f, \mathrm{d}g), \quad (f,g) = R(\mathrm{d}f, \mathrm{d}g), \quad \forall f,g \in C^\infty(\mathcal{O}^*).$$

Proposition 1. *The set* $\mathcal{F}_{\mathcal{O}}(\mathcal{O}^*)$ *is a* **Lie-Jordan algebra** *with products*

$$\{f_A, f_B\} = f_{[\![A,B]\!]}, \quad (f_A, f_B) = f_{A\odot B}, \quad A, B \in \mathcal{O}.$$

Hamiltonian and gradient vector fields on $\mathcal{O}^*$ are defined as:

$$X_f = -\Lambda(\mathrm{d}f, \cdot), \quad Y_f = R(\mathrm{d}f, \cdot), \quad f \in C^\infty(\mathcal{O}^*).$$

Unitary evolution of quantum systems is governed by a Hamiltonian vector field:

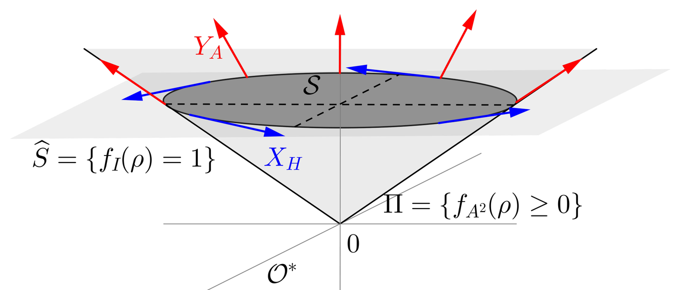
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = X_{f_H}(\rho(t)).$$

The set of states

The definition of the set $\mathcal{S}$ of states can be rewritten as

$$\mathcal{S} = \{\rho \in \mathcal{O}^* \mid f_I(\rho) = 1;\ f_{A^2}(\rho) \geq 0, \forall A \in \mathcal{O}\} \subset \mathcal{O}^*.$$

The set of states $\mathcal{S}$ is the **bounded** set of **positive** elements inside the hyperplane $\widehat{S}$ of **normalised** elements in $\mathcal{O}^*$. It is thus a manifold with boundary. Differential calculus is carried out by considering the **embedding** of $\mathcal{S}$ into the larger, differentiable manifold $\widehat{S}$. Hamiltonian and gradient vector fields on $\mathcal{O}^*$ preserve the rank of the elements [3], but not the normalization condition.



The set $\mathcal{S}$ is a stratified manifold [3,4], with the decomposition

$$\mathcal{S} = \bigcup_{j=1}^{n} \mathcal{S}_j, \quad \mathcal{S}_j \bigcap \mathcal{S}_k = \emptyset, \quad j \neq k.$$

with each stratum $\mathcal{S}_j$ the subset of states whose associated density matrix has rank j. Each stratum is on its own a differentiable manifold.

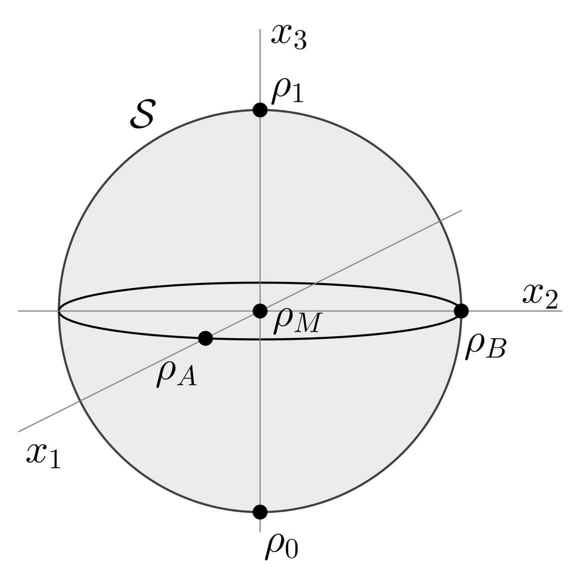
The set of states of a 2-level system is called the Bloch ball.

$$\rho = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & 1 - x_3 \end{pmatrix}, \quad x_1^2 + x_2^2 + x_3^2 \leq 1$$

$$\rho_0 = |0\rangle\langle 0|,\ \rho_1 = |1\rangle\langle 1|,\ \rho_M = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1$$

The algebra of observables, as linear functions, is $\mathfrak{su}(2)$

$$\{x_j, x_k\} = \sum_{l=1}^{3} \epsilon_{jkl} x_l \qquad j,k = 1,2,3$$
$$(x_j, x_k) = \delta_{jk}$$



Geometric description of the space of quantum states

Theorem 1. [2] *Consider an algebra* $(C^\infty(M), *)$ *of smooth functions on the manifold* M. *If the set* $\mathfrak{G}$ *of* **invariant functions** *with respect to the Lie group action* $\phi: G \times M \to M$ *is a* **subalgebra**, *then the* **restriction** *of the composition law* $*$ *to* $\mathfrak{G}$ *defines an* **algebraic structure** *on the set of smooth functions on the orbit set* M/G.

Consider the group action on $\mathcal{O}_0^* = \mathcal{O}^* - \{0\}$ generated by the dilation vector field $\Delta = Y_I$:

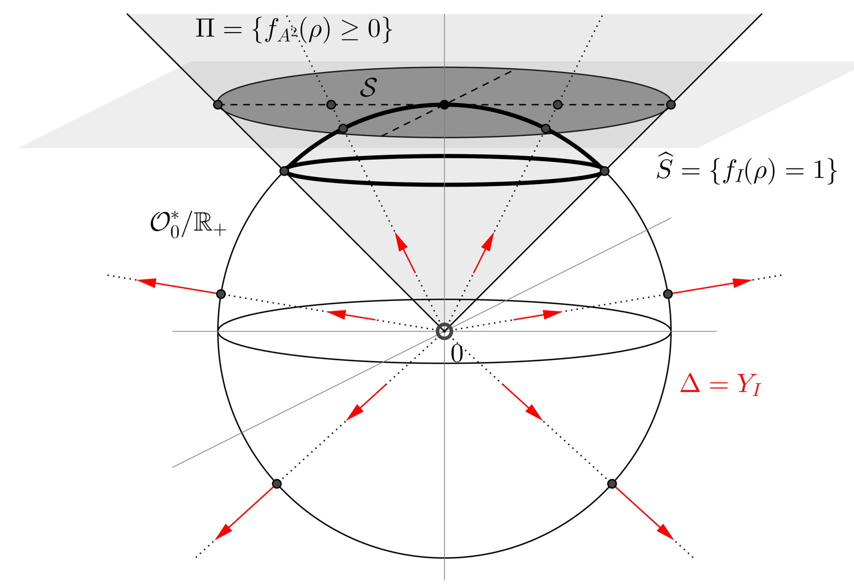
$$\phi: \mathbb{R}_+ \times \mathcal{O}_0^* \to \mathcal{O}_0^*, \quad \phi(a, \xi) = a\xi.$$

The quotient manifold $\mathcal{O}_0^*/\mathbb{R}_+$ can be embedded as the unit sphere in $\mathcal{O}_0^*$. The geometric objects cannot be projected by $\pi: \mathcal{O}_0^* \to \mathcal{O}_0^*/\mathbb{R}_+$, as they are not constant along the integral curves of Δ:

$$\mathcal{L}_\Delta \Lambda \neq 0, \quad \mathcal{L}_\Delta R \neq 0, \quad \mathcal{L}_\Delta f_A \neq 0$$

By relating points in the same orbits of the group action, it is possible to define a bijection

$$\varpi: (\mathcal{O}_0^*/\mathbb{R}_+) \cap \Pi \to \mathcal{S}$$



Invariant functions under the group action are expectation value functions of observables:

$$e_A(\xi) = \frac{f_A(\xi)}{f_I(\xi)} \Rightarrow e_A(a\xi) = e_A(\xi), \quad \forall a \in \mathbb{R}_+, \quad \forall \xi \in \mathcal{O}^*.$$

Expectation value functions and their products do not close any algebra with respect to the previously defined products:

$$\{e_A, e_B\} = \frac{1}{f_I} e_{[\![A,B]\!]}, \quad (e_A, e_B) = \frac{1}{f_I}(e_{A\odot B} - 2e_A e_B).$$

Theorem 2. *The set of expectation value functions on* $\mathcal{S}$ *is a* **Lie-Jordan algebra** *with respect to the products*

$$\{\epsilon_A, \epsilon_B\}_{\mathcal{S}} := \Lambda_{\mathcal{S}}(\mathrm{d}\epsilon_A, \mathrm{d}\epsilon_B) = \epsilon_{[\![A,B]\!]},$$
$$(\epsilon_A, \epsilon_B)_{\mathcal{S}} := R_{\mathcal{S}}(\mathrm{d}\epsilon_A, \mathrm{d}\epsilon_B) + 2\epsilon_A\epsilon_B = \epsilon_{A\odot B}.$$

with $e_A|_\Pi = (\varpi \circ \pi)^*(\epsilon_A)$, $\Lambda_{\mathcal{S}} = (\varpi \circ \pi)_*((f_I\Lambda)|_\Pi)$, $R_{\mathcal{S}} = (\varpi\circ\pi)_*((f_I\widehat{R})|_\Pi)$.

Relevant remarks

- The reduction procedure **breaks linearity**, in particular in the case of the Jordan structure. Non-linear behaviours are expected.
- **Unitary dynamics** is again generated by means of a Hamiltonian vector field, obtained by means of the Poisson tensor field $\Lambda_{\mathcal{S}}$ on the manifold of states.
- Tensor fields $\Lambda_{\mathcal{S}}$ and $R_{\mathcal{S}}$ are invariant under unitary dynamics. This is no longer true under more generic evolutions, which may lead to **contractions of the algebra of observables** [1,5].
- The action of $R_{\mathcal{S}}$ on expectation value function gives the deviation of the **Jordan product** from the **point-wise product**, i.e. of the non-local product with respect to the local product:

  $$R_{\mathcal{S}}(\mathrm{d}\epsilon_A, \mathrm{d}\epsilon_B)(\rho) = (\epsilon_A, \epsilon_B)_{\mathcal{S}}(\rho) - 2\epsilon_A(\rho)\epsilon_B(\rho), \quad A, B \in \mathcal{O}, \quad \rho \in \mathcal{S}.$$

  Furthermore, the tensor field $R_{\mathcal{S}}$ is related with the definitions of **variance** $\mathrm{Var}(A)$ and **covariance** $\mathrm{Cov}(A,B)$ of observables:

  $$R_{\mathcal{S}}(\mathrm{d}\epsilon_A, \mathrm{d}\epsilon_A)(\rho) = 2\epsilon_{A^2}(\rho) - 2(\epsilon_A(\rho))^2 = 2\mathrm{Var}(A)(\rho),$$
  $$R_{\mathcal{S}}(\mathrm{d}\epsilon_A, \mathrm{d}\epsilon_B)(\rho) = \epsilon_{A\odot B}(\rho) - 2\epsilon_A(\rho)\epsilon_B(\rho) = 2\mathrm{Cov}(A,B)(\rho),$$

  $R_{\mathcal{S}}$ is the main object reflecting the **quantum nature** of the system.

Markovian dynamics of open quantum systems

Markovian dynamics of open quantum systems is governed by the Kossakowski-Lindblad equation:

$$\frac{d}{dt}(\rho) = L(\rho) = -\mathrm{i}[H,\rho] - \frac{1}{2}\sum_{j=1}^{n^2-1}[V_j^\dagger V_j, \rho]_+ + \sum_{j=1}^{n^2-1} V_j \rho V_j^\dagger,$$

Theorem 3. *There exists a unique vector field* $Z_L \in \mathfrak{X}(\mathcal{S})$, *whose action on expectation value functions is*

$$Z_L(\epsilon_A)(\rho) = \epsilon_{L^\sharp(A)}(\rho), \quad \rho \in S,\ A \in \mathcal{O},$$

and whose integral curves are solutions to the Kossakowski-Lindblad equation.

Phase damping of a 2-level system:

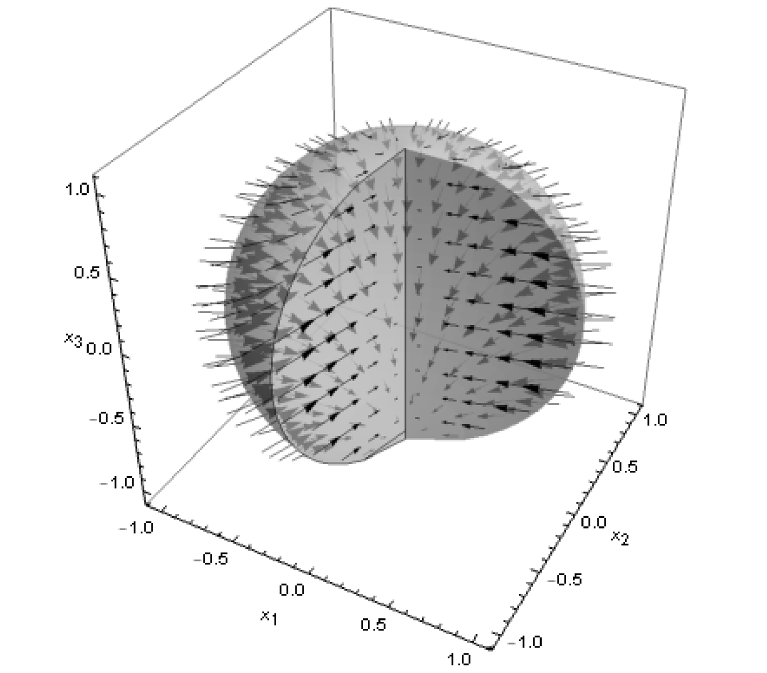
$$L(\rho) = -\gamma(\rho - \sigma_z\rho\sigma_z)$$

Tensor fields are not preserved by evolution:

$$\Lambda_{\mathcal{S},t} = e^{-t\mathcal{L}_{Z_L}}\Lambda_{\mathcal{S}}, R_{\mathcal{S},t} = e^{-t\mathcal{L}_{Z_L}} R_{\mathcal{S}}$$

The contracted algebra of observables in the limit $t \to \infty$ is the Euclidean algebra $\mathfrak{e}(2)$

$$\{x_1, x_2\}_\infty = 0,\ \{x_1, x_3\}_\infty = -x_2,\ \{x_2, x_3\}_\infty = x_1,$$
$$(x_1, x_1)_\infty = (x_2, x_2)_\infty = 0,\ (x_3, x_3)_\infty = 1.$$



Unitary controls in Markovian dynamics

Consider a controlled Markovian dynamics $Z_L + u_j(t)X_j$, with control functions $u_j(t)$ and Hamiltonian vector fields X_j. An accessible control system is **almost controllable** if it is possible to reach any neighbourhood of any point $\mathcal{S}$ in finite time.

Proposition 2. *An accessible open quantum system evolving under Markovian evolution whose limit manifold is a single pure state* ρ_L *is* **almost controllable by unitary controls**.

Proof. For the 2-level system, as seen in the Bloch Ball, all the points in the boundary are reachable from the limit point ρ_L by unitary controls. Integral curves of the Kossakowski-Lindblad vector field with no Hamiltonian term are straight lines [5]. Hence the 2-level system is controllable.

For an n-level system, take any state ρ_B on the boundary of $\mathcal{S}$. It has rank at most $n-1$, and it is either a pure state or a convex combination of a pure state ρ_B' and some other state. Pure states are almost reachable from ρ_L, as they are a leaf of the foliation generated by Hamiltonian vector fields. If ρ_B is not pure, unitary controls allow to almost reach ρ_B'. By induction, if the $(n-1)$-level system is almost reachable, then any state ρ_B on the boundary can be almost reached. Thus, the n-level system is almost controllable. □

Theorem 4. *Consider an open quantum system evolving under Markovian evolution and whose limit manifold is a subset of the boundary of the manifold of states. Then, the system is* **almost controllable by unitary controls**.

Proof. Let S_L be the limit manifold, and choose a pure state $\rho_L \in S_L$. It is possible to design a Hamiltonian vector field on $\mathcal{S}$ such that every other element in $\mathcal{S}_L$ is not stable. Thus, by controlling the system with this vector field, the problem reduces to the one presented in Proposition 2, with $\{\rho_L\}$ as the limit manifold, thus proving almost-controllability. □

References

[1] J. F. Cariñena, J. Clemente-Gallardo, J. A. Jover-Galtier, and G. Marmo. Tensorial dynamics on the space of quantum states. *Journal of Physics A: Mathematical and Theoretical*, 50(36):365301, 2017.

[2] F. Falceto, L. Ferro, A. Ibort, and G. Marmo. Reduction of Lie-Jordan algebras: Classical. *Il Nuovo Cimento C*, 36(3):107–115, 2013.

[3] J. Grabowski, M. Kuś, and G. Marmo. Geometry of quantum systems: density states and entanglement. *Journal of Physics A: Mathematical and General*, 38(47):10217–10244, 2005.

[4] J. Grabowski, M. Kuś, and G. Marmo. Symmetries, group actions, and entanglement. *Open Systems & Information Dynamics*, 13:343–362, 2006.

[5] J. A. Jover-Galtier. *Sistemas cuánticos abiertos: descripción geométrica, dinámica y control.* PhD thesis, Universidad de Zaragoza, 2017.