Rolling Grassmannians – Constructive Proof of Controllability

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Motivation Research Areas for Study the Problem and Goals (F. Pina, F. Silva Leite '18, A. Srivastava, P. Turaga '16, R. Vemulapalli, R. Chellapa '16)

- Robotics and Computer vision
- Engineering problems that deals with:
  - Set of images
  - Face recognition
  - Reconstruction of planar scenes from multiples views
- Medical Engineering applications

- The Grassmann manifolds (Grassmannians) can be widely used to represent images
- Describe a pure rolling of Grassmannians over their affine tangent space at a particular point
- Show how the forbidden motions of twist and slip can be accomplished by rolling without them

Some Background

- Matrix representation of Grassmannians

$$G_{k,n} := \{P \in \mathfrak{s}_n : \ P^2 = P, \ \mathrm{rank}(P) = k\}$$

- Tangent space at a point $P \in G_{k,n}$

$$\begin{aligned} T_P G_{k,n} &= \{S \in \mathfrak{s}_n : \ PS + SP = S\} \\ &= \{[\Omega, P] : \ \Omega \in \mathfrak{so}_n, P\Omega + \Omega P = \Omega\} \end{aligned}$$

$G_{k,n}$ is an isospectral manifold and for $P_0 = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$

$$T_{P_0} G_{k,n} = \left\{ \begin{bmatrix} 0 & Z \\ Z^\top & 0 \end{bmatrix}, \quad Z \in \mathbb{R}^{k \times (n-k)} \right\}$$

Rolling Grassmann over the affine tangent space at P_0

- $\overline{M} = \mathfrak{s}_n; \qquad \overline{G} = \mathrm{SO}_n \ltimes \mathfrak{s}_n$
  ($\overline{M}$ embedding manifold; $\overline{G}$ isometry group of $\overline{M}$)
- The action of $\overline{G}$ on $\overline{M}$ is defined by

$$((\Theta, X), S) \longmapsto \Theta S \Theta^\top + X$$

- $M_1 = G_{k,n}; \qquad M_0 = T^{\mathrm{aff}}_{P_0} G_{k,n} := P_0 + T_{P_0} G_{k,n}$
  (M_1 rolling manifold; M_0 static manifold)

Theorem (Jurdjevic and Sussmann, 1972)

A left-invariant control system without drift and unrestricted controls, evolving on a connected Lie group G, is controllable if and only if the control vector fields generate $\mathcal{L}(G)$, the Lie algebra of G, i.e., satisfy the bracket generating property.

Kinematic equations of rolling $G_{k,n}$ over $T^{\mathrm{aff}}_{P_0} G_{k,n}$ (Hüpper and Silva Leite, '07, F. Pina, F. Silva Leite '18)

$$\begin{cases} \dot{\Theta}(t) = \Theta(t) A(t) \\ \dot{X}(t) = B(t) \end{cases}, \quad A(t) := \begin{bmatrix} 0 & -U(t) \\ U^\top(t) & 0 \end{bmatrix}, \quad B(t) := \begin{bmatrix} 0 & U(t) \\ U^\top(t) & 0 \end{bmatrix}, \qquad (\ t \mapsto U(t) \in \mathbb{R}^{k\times(n-k)} \text{ is free})$$

- Can be rewritten as a left-invariant control system without drift, evolving on the connected Lie group $G = \mathrm{SO}_n \times T_{P_0} G_{k,n}$, with Lie algebra $\mathcal{L}(G) = \mathfrak{so}_n \oplus T_{P_0} G_{k,n}$.
- Left-invariant control vector fields $(A_{i,k+j}, B_{i,k+j})$, $i = 1, \cdots, k$, $j = 1, \cdots, n-k$, with $A_{i,j} = E_{i,j} - E_{j,i}$ (elementary skew symmetric matrices), $B_{i,j} = E_{i,j} + E_{j,i}$ (elementary symmetric matrices) satisfy the bracket generating property.
- Whenever $k(n-k) \neq 1$, the control system (kinematic equations) is controllable in $G = \mathrm{SO}_n \times T_{P_0} G_{k,n}$.

Forbidden motion - Slip

Any motion of $G_{k,n}$ that results from the action of elements in $\overline{G}$ of the form (I_n, X), where $X \in T_{P_0} G_{k,n}$.

(A pure translation by a vector X)

Forbidden motion - Twist

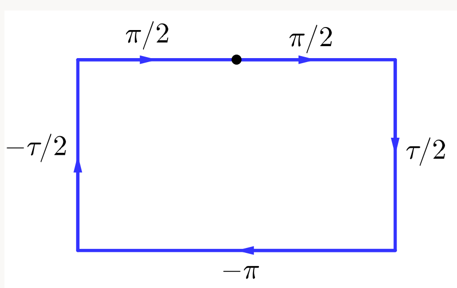
Any motion of $G_{k,n}$ that results from the action of elements in $\overline{G}$ of the form $(\Theta, 0)$, where $\Theta \in K$, the isotropy subgroup of SO_n at P_0.

(A pure rotation that fixes P_0)

How to generate the forbidden motions of slip and twist, without twisting and without slipping?

Generating Twists (F. Pina, F. Silva Leite '18, M. Kleinsteuber, K. Hüper, F. Silva Leite '06)

Crucial Result



If A, B and C are square matrices s.t. $[A, B] = C, \quad [A, C] = -B$, then

$$e^{\tau C} = e^{(\pi/2)A}\, e^{(\tau/2)B}\, e^{-\pi A}\, e^{-(\tau/2)B}\, e^{(\pi/2)A}.$$

Every $\Theta \in K$ can be written as a finite product of elements of the form

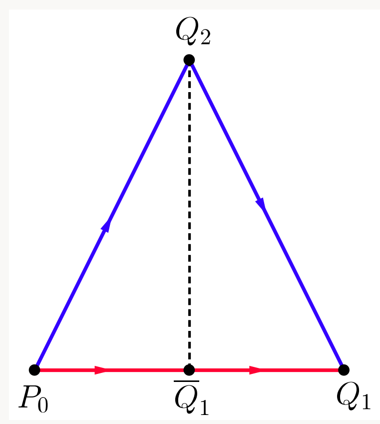
$$\left[\begin{array}{c|c} e^{\tau_1 A_{i,j}} & 0 \\ \hline 0 & e^{\tau_2 A_{k+l,k+m}} \end{array}\right], \quad {\scriptstyle 1 \le i < j \le k \atop 1 \le l < m \le n-k}.$$

- $e^{\tau_1 A_{i,j}}$ and $e^{\tau_2 A_{k+l,k+m}}$ can be decomposed into products of Givens rotations generated by elements of the form $A_{r,k+s}$, $r = 1, \cdots, k$, $s = 1, \cdots, n-k$, so that the sum of all angles of rotation adds up to zero:

$$\begin{aligned} e^{\tau_1 A_{i,j}} &= e^{(\pi/2)A_{j,k+l}}\, e^{(\tau_1/2)A_{i,k+l}}\, e^{-\pi A_{j,k+l}} e^{-(\tau_1/2)A_{i,k+l}}\, e^{(\pi/2)A_{j,k+l}} \\ e^{\tau_2 A_{k+l,k+m}} &= e^{(\pi/2)A_{i,k+m}}\, e^{(\tau_2/2)A_{i,k+l}}\, e^{-\pi A_{i,k+m}} e^{-(\tau_2/2)A_{i,k+l}}\, e^{(\pi/2)A_{i,k+m}}. \end{aligned}$$

Generating Slips (F. Pina, F. Silva Leite '18, M. Kleinsteuber, K. Hüper, F. Silva Leite '06)

Let $X = \tau B_{1,k+1}, \tau > 0$. To perform a slip from P_0 to $Q_1 = P_0 + X$ in $T^{\mathrm{aff}}_{P_0} G_{k,n}$, there are two situations to consider, depending on the distance between the points: $d(P_0, Q_1) = \tau\sqrt{2}$.



- 1^{st}: τ is a multiple of 2π
  - Pure rolling of $G_{k,n}$ along a geodesic arc, so that its development curve is the geodesic arc in $T^{\mathrm{aff}}_{P_0} G_{k,n}$ that joins P_0 to Q_1.
- 2^{nd}: τ is not a multiple of 2π
  - Choose $Q_2 = \overline{Q}_1 + \tau_1 B_{1,k+2} = P_0 + \frac{\tau}{2} B_{1,k+1} + \tau_1 B_{1,k+2}$, where $\overline{Q}_1$ is the midpoint between P_0 and Q_1 and τ_1 must be chosen so that $d(P_0, Q_2) = d(Q_1, Q_2) = 2\pi r\sqrt{2}, (r \in \mathbb{N})$
  - Construct an isosceles triangle as shown in the picture, and repeat the procedure of 1^{st} along the two blue sides of the triangle.

References

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