Affine Lagrangians in the k-cosymplectic formulation of field theories

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13th International Young Researchers Workshop on Geometry, Mechanics and Control

University of Coimbra, Coimbra, Portugal • 6-8 December 2018

ABSTRACT Field theories described by Lagrangians depending on space-time coordinates can be modelled using k-cosymplectic geometry. In particular, theories described by singular Lagrangians are of special interest because of their role in modern physics, in particular in gauge field theories. Nevertheless, there is a problem of consistency of the corresponding equations of motion (Euler–Lagrange and Hamilton–de Donder–Weyl), and thus one needs to study the possible constraints arising in the space of solutions. This problem can be analyzed by means of a constraint algorithm. On the other hand, some field theories are based on affine Lagrangians, i. e., Lagrangians that are affine with respect to the velocity variables. We apply the constraint algorithm to this specific case, both in the Lagrangian and the Hamiltonian formalisms.

k-PRECOSYMPLECTIC MANIFOLDS

Definition

*Let $\mathcal{M}$ be a differentiable manifold of dimension $n(k+1)+k-m$ (with $1\leq m\leq nk$). A k-**precosymplectic structure** in $\mathcal{M}$ is a family $(\eta^\alpha,\Omega^\alpha,\mathcal{V})$, $1\leq\alpha\leq k$, where η^α are closed 1-forms in $\mathcal{M}$, Ω^α are closed 2-forms in $\mathcal{M}$, and $\mathcal{V}$ is an integrable nk-dimensional integrable distribution in $\mathcal{M}$ satisfying that:*

(1) $\eta^1\wedge\cdots\wedge\eta^k\neq 0,\quad \eta^\alpha|_{\mathcal{V}}=0,\quad \Omega^\alpha|_{\mathcal{V}\times\mathcal{V}}=0,$

(2) $\left(\bigcap_{\alpha=1}^{k}\ker\eta^\alpha\right)\cap\left(\bigcap_{\alpha=1}^{k}\ker\Omega^\alpha\right)\neq\{0\}\ ,\ \ \mathrm{rank}\left(\bigcap_{\alpha=1}^{k}\ker\Omega^\alpha\right)>k.$

*A manifold $\mathcal{M}$ endowed with a k-precosymplectic structure is called a k-**precosymplectic manifold**.*

This definition is a generalization of the definition of k-cosymplectic manifold [1, 2] to model the geometry of singular nonautonomous field theories. Now we are going to describe a particular kind of local coordinates, that we will call **Darboux coordinates**.

Theorem (k-precosymplectic Darboux theorem)

Let $\mathcal{M}$ be a k-precosymplectic manifold such that $\mathrm{rank}\,\Omega^\alpha=2r_\alpha$, with $1\leq r_\alpha\leq n$ and $m=kn-\sum_{\alpha=1}^{k}r_\alpha-d$. For every point $p\in\mathcal{M}$, there exists a local chart

$$(\mathcal{U}_p;x^\alpha,y^i,y^\alpha_{i_\alpha},z^\alpha_{j_\alpha})\quad(1\leq\alpha\leq k\ ,\ 1\leq i\leq n\ ,\ i_\alpha\in I_\alpha\subseteq\{1,\dots,n\}\ ,\ 1\leq j\leq d),$$

such that

$$\eta^\alpha|_{\mathcal{U}_p}=\mathrm{d}x^\alpha\quad,\qquad \Omega^\alpha|_{\mathcal{U}_p}=\mathrm{d}y^{i_\alpha}\wedge\mathrm{d}y^\alpha_{i_\alpha}$$
$$\mathcal{V}|_{\mathcal{U}_p}=\left\langle\frac{\partial}{\partial y^\alpha_{i_\alpha}},\frac{\partial}{\partial z^\alpha_{j_\alpha}}\right\rangle\quad,\qquad\left[\left(\bigcap_{\alpha=1}^{k}\ker\eta^\alpha\right)\cap\left(\bigcap_{\alpha=1}^{k}\ker\Omega^\alpha\right)\right]\Bigg|_{\mathcal{U}_p}=\left\langle\frac{\partial}{\partial z^\alpha_{j_\alpha}}\right\rangle\ .$$

The proof of this theorem will be found in [3].

Proposition

Given a k-precosymplectic manifold $(M,\omega^\alpha,\eta^\alpha,V)$, there exists a family $\mathcal{R}_1,\dots,\mathcal{R}_k\in\mathfrak{X}(M)$ of vector fields satisfying

$$\begin{cases}i_{\mathcal{R}_\alpha}\omega^\beta=0,\\ i_{\mathcal{R}_\alpha}\eta^\beta=\delta^\beta_\alpha.\end{cases}\tag{1}$$

Such a family of vector fields is called a family of **Reeb vector fields**.

k-PRECOSYMPLECTIC HAMILTONIAN SYSTEMS

Definition

*A k-**precosymplectic Hamiltonian system** is a family $(M,\omega^\alpha,\eta^\alpha,V,\gamma)$ where $(M,\omega^\alpha,\eta^\alpha,V)$ is a k-precosymplectic manifold of the type $\mathbb{R}^k\times P$ and γ is a closed 1-form on M called the **Hamiltonian** 1-**form**.*

The solutions of a k-precosymplectic Hamiltonian system are the integral sections of the k-vector fields $\mathcal{X}=(X_\alpha)\in\mathfrak{X}^k(M)$ solution of the system of differential equations

$$\begin{cases}i_{X_\alpha}\omega^\alpha=\gamma-\gamma(\mathcal{R}_\alpha)\eta^\alpha,\\ i_{X_\alpha}\eta^\beta=\delta^\beta_\alpha.\end{cases}\tag{2}$$

k-PRECOSYMPLECTIC CONSTRAINT ALGORITHM

We want to find a submanifold $N\hookrightarrow M$ such that the previous system of equations has global solutions on N tangent to N. In order to find this submanifold (if it exists) we construct a constraint algorithm which provides us with a sequence of submanifolds

$$\cdots\hookrightarrow M_j\hookrightarrow\cdots\hookrightarrow M_2\hookrightarrow M_1\hookrightarrow M$$

which in favorable cases will end in a final constraint submanifold.

The following theorem will be the core of our algorithm and will give us a way to compute the constraints at every step of the algorithm.

Theorem

Let $(M,\omega^\alpha,\eta^\alpha,V,\gamma)$ be a k-precosymplectic Hamiltonian system. Consider a submanifold $C\hookrightarrow M$ and a k-vector field $\mathcal{X}\colon C\to(T^1_k)_CM$ such that $\mathcal{X}_p\in(T^1_k)_pC$ for every $p\in C$. The following two conditions are equivalent:

(1) *There exists a k vector field $\mathcal{X}=(X_\alpha)\colon C\to(T^1_k)_CM$ tangent to C such that the system of equations*

$$\begin{cases}i_{X_\alpha}\omega^\alpha=\gamma-\gamma(\mathcal{R}_\alpha)\eta^\alpha,\\ i_{X_\alpha}\eta^\beta=\delta^\beta_\alpha,\end{cases}$$

holds on C.

(2) *For every $p\in C$, there exists $\mathcal{Z}_p=(Z_\alpha)_p\in(T^1_k)_pC$ such that $i_{Z_{\alpha p}}\eta^\beta_p=\delta^\beta_\alpha$ and $\sum_\alpha\eta^\alpha_p+\widetilde{\gamma}_p=\flat(\mathcal{Z}_p)$, where $\widetilde{\gamma}_p=\gamma_p-\gamma_p(\mathcal{R}_{\alpha p})\eta^\alpha_p$ and $\flat$ is defined as*

$$\begin{array}{rccl}\flat\colon & T^1_kM & \longrightarrow & T^*M\\ & \mathcal{X}=(X_\alpha) & \longmapsto & i(X_\alpha)\omega^\alpha+(i(X_\alpha)\eta^\alpha)\eta^\alpha.\end{array}$$

Taking into account the previous theorem, we define the j-ary constraint submanifold as

$$M_j=\left\{p\in M_{j-1}\,|\,\exists\mathcal{Z}=(Z_\alpha)\in(T^1_k)M_{j-1}\text{ such that }\flat(\mathcal{Z})=\widetilde{\gamma}+\sum_\alpha\eta^\alpha\text{ and }i_{Z_\alpha}\eta^\beta=\delta^\beta_\alpha\right\},$$

where $M_0=M$.

Definition

*Let $C\hookrightarrow M$ be a submanifold of a k-precosymplectic manifold M. The k-**precosymplectic orthogonal complement** of C is*

$$TC^\perp=\left(\flat\left((T^1_k)C\cap D_C\right)\right)^0$$

where D_C is the set of all k-vectors $\mathcal{Z}_p=(Z_\alpha)_p$ on C such that $i_{Z_{\alpha p}}\eta^\beta_p=\delta^\beta_\alpha$.

Using this definition we can characterize the ***j*-ary constraint submanifold** as

$$M_j=\left\{p\in M_{j-1}\,|\,\widetilde{\gamma}+\sum_\alpha\eta^\alpha\in((TC)^\perp)^0\right\}.$$

Although this allows us to effectively compute the constraints at every step of the algorithm, an alternative and equivalent way to compute the constraint submanifolds of the ***k*-precosymplectic constraint algorithm**, which is much more operational, is the following:

(1) Obtain a local basis $\{Z_1,\dots,Z_r\}$ of $(TM)^\perp$.
(2) Use the theorem to obtain a set of independent constraint functions $f_\mu=i(Z_\mu)(\widetilde{\gamma}+\sum_\alpha\eta^\alpha)$ defining the submanifold $M_1\hookrightarrow M$.
(3) Compute solutions $\mathcal{X}=(X_\alpha)$ of the field equations on M.
(4) Impose the tangency condition of $X_1,\dots,X_k$ on M_1.
(5) Iterate item (4) until no new constraints appear.

If this iterative procedure ends in a non-empty submanifold M_f, then we can assure the existence of global solutions to the field equations on this submanifold M_f.

Remark Let $\mathcal{L}$ be a singular nonautonomous Lagrangian on $\mathbb{R}^k\times T^1_kQ$. It can be modelled as the k-precosymplectic Hamiltonian system $(\mathbb{R}^k\times T^1_kQ,\omega^\alpha_{\mathcal{L}},\mathrm{d}x^\alpha,\mathrm{d}E_{\mathcal{L}})$. Therefore, we can apply the k-precosymplectic constraint algorithm to it.

AFFINE LAGRANGIANS

Some field theories of interest in theoretical physics are modelled on affine Lagrangians; this is the case, for instance, of Einstein–Palatini gravity or Dirac fermion fields. We are going to apply the constraint algorithm to such theories, both in the Lagrangian and Hamiltonian formalisms [4].

Consider $\pi_{\mathbb{R}^k}\colon\mathbb{R}^k\times Q\to\mathbb{R}^k$ as the configuration bundle of a field theory and its associated phase space bundle of k-velocities $\bar{\tau}_1\colon\mathbb{R}^k\times T^1_kQ\to\mathbb{R}^k$, with coordinates $(x^\alpha,q^i,v^i_\alpha)$. In this phase space we consider an **affine Lagrangian**, that is, a Lagrangian function $L\in C^\infty(\mathbb{R}^k\times T^1_kQ)$ affine in the fibre coordinates v^i_α:

$$L(x^\alpha,q^i,v^i_\alpha)=f^\mu_j(x^\alpha,q^i)v^j_\mu+g(x^\alpha,q^i)\ .\tag{3}$$

Obviously such a Lagrangian is singular.

Remark An affine Lagrangian can be alternatively defined from a 2-semibasic k-form ζ on $\mathbb{R}^k\times Q$. From it a Lagrangian $L_\zeta\in C^\infty(\mathbb{R}^k\times T^1_kQ)$ is determined by the equality

$$L_\zeta(x,j^1_0\phi_x)\,\omega_x:=[\phi^*\zeta]\,(x)\ ,$$

where ϕ is any section of $\mathbb{R}^k\times Q\to\mathbb{R}^k$ and $\omega=\mathrm{d}^kx$ is the volume form of $\mathbb{R}^k$. This function is well defined and its local expression is that of an affine Lagrangian.

LAGRANGIAN FORMALISM

Now let us reproduce the calculations for an affine Lagrangian. We have

$$\begin{aligned}E_L\ &=\ \Delta(L)-L=-g(x^\alpha,q^i)\in C^\infty(\mathbb{R}^k\times T^1_kQ)\ ,\\ \omega^\alpha_L\ &=\ -\left(\frac{\partial f^\alpha_k}{\partial x^\mu}\mathrm{d}x^\mu+\frac{\partial f^\alpha_k}{\partial q^j}\mathrm{d}q^j\right)\wedge\mathrm{d}q^k\in\Omega^2(\mathbb{R}^k\times T^1_kQ)\ ,\end{aligned}$$

and we have a k-precosymplectic structure $(\omega^\alpha_L,\mathrm{d}x^\alpha,\mathcal{V})$ in $\mathbb{R}^k\times T^1_kQ$ where (x^α) are the coordinates of $\mathbb{R}^k$, $\mathcal{V}=\left\langle\frac{\partial}{\partial v^i_\mu}\right\rangle$ and where the Reeb vector fields, defined in (1), are $\mathcal{R}_\alpha=\frac{\partial}{\partial x^\alpha}$. Then

$$\mathrm{d}E_L+\frac{\partial L}{\partial x^\mu}\mathrm{d}x^\mu=-\frac{\partial g}{\partial x^\mu}\mathrm{d}x^\mu-\frac{\partial g}{\partial q^j}\mathrm{d}q^j+\left(\frac{\partial f^\nu_l}{\partial x^\mu}v^l_\nu+\frac{\partial g}{\partial x^\mu}\right)\mathrm{d}x^\mu=-\frac{\partial g}{\partial q^j}\mathrm{d}q^j+\frac{\partial f^\nu_l}{\partial x^\mu}v^l_\nu\,\mathrm{d}x^\mu\ ,$$

and, for a k-vector field $\mathcal{X}=(X_1,\dots,X_k)\in\mathfrak{X}^k(\mathbb{R}^k\times T^1_kQ)$ satisfying the second group of equations (2), we have that

$$X_\alpha=\frac{\partial}{\partial x^\alpha}+F^l_\alpha\frac{\partial}{\partial q^l}+G^l_{\alpha\nu}\frac{\partial}{\partial v^l_\nu}\in\mathfrak{X}(\mathbb{R}^k\times T^1_kQ)\ ,\tag{4}$$

thus

$$i_{X_\alpha}\omega^\alpha_L=F^l_\alpha\frac{\partial f^\alpha_l}{\partial x^\mu}\mathrm{d}x^\mu-\frac{\partial f^\alpha_j}{\partial x^\alpha}\mathrm{d}q^j+F^l_\alpha\left(\frac{\partial f^\alpha_l}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial q^l}\right)\mathrm{d}q^j\ ,$$

and the first group of equations (2) leads to

$$(v^l_\nu-F^l_\nu)\frac{\partial f^\nu_l}{\partial x^\mu}\ =\ 0\ ,\tag{5}$$
$$\frac{\partial g}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial x^\alpha}\ =\ -F^l_\alpha\left(\frac{\partial f^\alpha_l}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial q^l}\right)\ .\tag{6}$$

This is a system of (linear) equations for the component functions F^l_α, which allows us to determine (partially) these functions and, eventually, gives raise to constraints functions (depending on the rank of the matrices involved). If this last situation happens, then the constraint algorithm follows by demanding the tangency condition for the vector fields X_α. Observe also that, in any case, in these vector fields, the coefficients $G^i_{\alpha\nu}$ are undetermined.

If we look for semi-holonomic k-vector fields $\mathcal{X}$, it implies that $F^k_\nu=v^k_\nu$ in (4). Then, equations (5) hold identically, meanwhile equations (6) read

$$\frac{\partial g}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial x^\alpha}+v^l_\alpha\left(\frac{\partial f^\alpha_l}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial q^l}\right)=0$$

which are constraints. Then the tangency condition for the vector fields

$$X_\nu=\frac{\partial}{\partial x^\nu}+v^l_\nu\frac{\partial}{\partial q^l}+G^l_{\nu\alpha}\frac{\partial}{\partial v^l_\alpha}\ ,$$

leads to

$$\frac{\partial g}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial x^\alpha}+G^l_{\nu\alpha}\left(\frac{\partial f^\alpha_l}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial q^l}\right)=0$$

which allows us to determine (partially) the functions $G^k_{\nu\alpha}$ and, eventually, gives raise to constraints functions, depending on the rank of the matrix $\left(\frac{\partial f^\alpha_l}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial q^l}\right)$. In this last case, the constraint algorithm continues by demanding again the tangency condition.

HAMILTONIAN FORMALISM

Let $\bar{\pi}_1\colon\mathbb{R}^k\times(T^1_k)^*Q\to\mathbb{R}^k$ be the phase space bundle of k-momenta. The Legendre map $\mathcal{F}L\colon\mathbb{R}^k\times T^1_kQ\to\mathbb{R}^k\times(T^1_k)^*Q$ associated to the Lagrangian L is

$$x^\mu\circ\mathcal{F}L=x^\mu\quad,\quad q^i\circ\mathcal{F}L=q^i\quad,\quad p^\mu_i\circ\mathcal{F}L=\frac{\partial L}{\partial v^i_\mu}=f^\mu_i(x^\alpha,q^j)\ .$$

Observe that $\mathcal{P}=\mathcal{F}L(\mathbb{R}^k\times T^1_kQ)$ is defined by the constraints $p^\mu_i=f^\mu_i(x^\alpha,q^j)$; hence it is the image of a section $\xi\colon\mathbb{R}^k\times Q\to\mathbb{R}^k\times(T^1_k)^*Q$ of the projection $(\pi_Q)_{(1,0)}\colon\mathbb{R}^k\times(T^1_k)^*Q\to\mathbb{R}^k\times Q$, and then it can be identified in a natural way with $\mathbb{R}^k\times Q$. Therefore, as $\xi\circ\tau_1$ is a surjective submersion with connected fibres, then so is $\mathcal{F}L_0\colon\mathbb{R}^k\times T^1_kQ\to\mathcal{P}$ (the restriction of $\mathcal{F}L$ onto its image $\mathcal{P}$), since $\mathcal{F}L_0=\xi\circ\tau_1$. In conclusion, affine Lagrangians are almost regular Lagrangians and thus $\mathcal{P}$ is an embedded submanifold of $\mathbb{R}^k\times(T^1_k)^*Q$, which is diffeomorphic to $\mathbb{R}^k\times Q$.

Therefore we can introduce

$$\begin{aligned}h\ &=\ -g(x^\alpha,q^i)\in C^\infty(\mathcal{P})\ ,\\ \omega^\alpha\ &=\ -\left(\frac{\partial f^\alpha_k}{\partial x^\mu}\mathrm{d}x^\mu+\frac{\partial f^\alpha_k}{\partial q^j}\mathrm{d}q^j\right)\wedge\mathrm{d}q^k\in\Omega^2(\mathcal{P})\ ,\end{aligned}$$

such that $\mathcal{F}L_0^*E_L=h$ and $\mathcal{F}L_0^*\omega^\alpha_L=\omega^\alpha$. As above, $\eta^\alpha=\mathrm{d}x^\alpha$ and the Reeb vector fields are $\mathcal{R}_\alpha=\frac{\partial}{\partial x^\alpha}$. Then

$$\mathrm{d}h-\mathcal{R}_\alpha(\mathrm{d}h)\,\mathrm{d}x^\mu=\mathrm{d}h-\frac{\partial h}{\partial x^\mu}\mathrm{d}x^\mu=-\frac{\partial g}{\partial x^\mu}\mathrm{d}x^\mu-\frac{\partial g}{\partial q^j}\mathrm{d}q^j+\frac{\partial g}{\partial x^\mu}\mathrm{d}x^\mu=-\frac{\partial g}{\partial q^j}\mathrm{d}q^j\in\Omega^1(\mathcal{P})\ ,$$

and, for a k-vector field $\mathcal{X}=(X_1,\dots,X_k)\in\mathfrak{X}^k(\mathcal{P})$ satisfying the second group of equations (2), we have that

$$X_\alpha=\frac{\partial}{\partial x^\alpha}+F^l_\alpha\frac{\partial}{\partial q^l}\in\mathfrak{X}(\mathcal{P})\ ,$$

thus

$$i_{X_\alpha}\omega^\alpha=F^l_\alpha\frac{\partial f^\alpha_l}{\partial x^\mu}\mathrm{d}x^\mu-\frac{\partial f^\alpha_j}{\partial x^\alpha}\mathrm{d}q^j\ ,$$

and the first group of equations (2) leads to

$$F^l_\nu\frac{\partial f^\nu_l}{\partial x^\mu}=0\quad,\quad\frac{\partial g}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial x^\alpha}=-F^l_\alpha\left(\frac{\partial f^\alpha_l}{\partial q^j}-\frac{\partial f^\alpha_j}{\partial q^l}\right)\ .$$

As in the Lagrangian formalism, this system of (linear) equations allows us to determine (partially) the component functions F^l_α and, eventually, could give constraints functions (depending on the rank of the matrices involved). If this last situation happens, then the constraint algorithm follows by demanding the tangency condition for the vector fields X_α.

ACKNOWLEDGMENTS

We acknowledge the financial support from the Spanish Ministerio de Economía y Competitividad project MTM2014–54855–P and from the Catalan Government project 2017–SGR–932.

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