Lie symmetries for a multi-component NLS Equation in $2+1$ dimensions

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Outline

The study of symmetries represents a fundamental point related to the analysis of integrability of differential equations, since this invariance property may be used to achieve partial or complete integration of such equations.

Introduction

Symmetries and similarity reductions

The study of symmetries represents a fundamental point related to the analysis of integrability of differential equations, since this invariance property may be used to achieve partial or complete integration of such equations.

A standard method for finding solutions of a PDE can be implemented using Lie symmetries: each Lie symmetry leads to a similarity reduction for the PDE which allow us to reduce by one the number of variables.

In this work, we are concerned with the analysis of the Lax pair, considered as a proof of the integrability of a PDE. As mentioned, Lie symmetries for the PDE are very popular in literature, but the symmetry analysis for the Lax pair is often less studied.

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This approach has the benefit that the reduction associated to each symmetry of the Lax pair provides both the reduced equations and the reduced spectral problem.

Multi-component NLS Equation in $2+1$

P. Albares et al., arXiv:1807.09039v1 [nlin.SI]

An integrable multi-component nonlinear Schrödinger equation in $2+1$ dimensions is presented:

$$\begin{aligned} &i\vec{\alpha}_t + \vec{\alpha}_{xx} + 2\, m_x\, \vec{\alpha} = 0, \quad -i\vec{\alpha}^\dagger_t + \vec{\alpha}^\dagger_{xx} + 2\, m_x\, \vec{\alpha}^\dagger = 0, \\ &\left(m_y + \vec{\alpha}\,\vec{\alpha}^\dagger\right)_x = 0, \end{aligned} \tag{1}$$

where $\vec{\alpha}(x,y,t) = (\alpha_1(x,y,t),\ \alpha_2(x,y,t))^\intercal$ and $\vec{\alpha}^\dagger$ is the complex conjugate of $\vec{\alpha}$. $m(x,y,t)$ is a real scalar function related to the probability density $\vec{\alpha}\,\vec{\alpha}^\dagger$.

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- The reduction $x = y$ of (1) yields the Manakov system, also called vector NLS system.
- (1) is a multi-component generalization of the simplest NLS Eq. in $2+1$ dimensions.

Lax Pair

The three-component Lax pair for (1), and its complex conjugate, has the following form:

$$\begin{aligned} \psi_y &= -\alpha_1^{\dagger}\,\chi - \alpha_2^{\dagger}\,\rho, & \psi_t &= -\psi_{xx} - 2\,m_x\,\psi \\ \chi_x &= -\alpha_1\,\psi, & \chi_t &= -(\alpha_1)_x\,\psi + \alpha_1\,\psi_x \\ \rho_x &= -\alpha_2\,\psi, & \rho_t &= -(\alpha_2)_x\,\psi + \alpha_2\,\psi_x \end{aligned} \tag{2}$$

where the eigenvector of the Lax Pair is defined by $\vec{\Psi}(x,y,t) = (\psi(x,y,t),\ \chi(x,y,t),\ \rho(x,y,t))^{\mathsf{T}}$.

Note that no spectral parameter appears in this Lax Pair.

Lie point symmetries

The following infinitesimal Lie point symmetries have been considered:

$$\begin{aligned}
\hat{x} &= x + \varepsilon\,\xi_1(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{y} &= y + \varepsilon\,\xi_2(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{t} &= t + \varepsilon\,\xi_3(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{\alpha}_1 &= \alpha_1 + \varepsilon\,\eta_1(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{\alpha}_2 &= \alpha_2 + \varepsilon\,\eta_2(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{m} &= m + \varepsilon\,\eta_3(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{\psi} &= \psi + \varepsilon\,\phi_1(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{\chi} &= \chi + \varepsilon\,\phi_2(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2),\\
\hat{\rho} &= \rho + \varepsilon\,\phi_3(x,y,t,\alpha_1,\alpha_2,m,\psi,\chi,\rho) + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{3}$$

where ε is the group parameter and ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, ϕ_1, ϕ_2, ϕ_3 are the infinitesimals.

Lie point symmetries

The infinitesimal generator of the group of the previous transformations is given by the vector field

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial \omega} + \eta_3 \frac{\partial}{\partial m} + \phi_1 \frac{\partial}{\partial \psi} + \phi_2 \frac{\partial}{\partial \chi} + \phi_3 \frac{\partial}{\partial \rho} \tag{4}$$

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This infinitesimal transformation induces a well known one in the derivatives of the fields. This procedure, applied to (2), leads to an overdetermined system of PDEs for the infinitesimals, whose solution provides the symmetry transformations.

Results: Lie symmetries for (1)

The following sets of symmetries have been obtained:

$$
\begin{aligned}
\xi_1 &= 4\dot{K}_1(t)\,x + 2K_2(t) \\
\xi_2 &= 2C_1(y) \\
\xi_3 &= 8K_1(t) \\
\eta_1 &= \left[i\left(\ddot{K}_1(t)x^2 + \dot{K}_2(t)x + K_3(t) + C_2(y)\right) - 2\dot{K}_1(t) - C_1'(y)\right]\alpha_1 \\
&\quad + \left[C_4(y) + i\,C_5(y)\right]\alpha_2 \\
\eta_2 &= \left[i\left(\ddot{K}_1(t)x^2 + \dot{K}_2(t)x + K_3(t) + C_3(y)\right) - 2\dot{K}_1(t) - C_1'(y)\right]\alpha_2 \\
&\quad - \left[C_4(y) - i\,C_5(y)\right]\alpha_1 \\
\eta_3 &= -4\dot{K}_1(t)\,m + \frac{1}{6}\dddot{K}_1(t)x^3 + \frac{1}{4}\ddot{K}_2(t)x^2 + \frac{1}{2}\dot{K}_3(t)x + \delta(y,t)
\end{aligned}
\tag{5}
$$

Results: Lie symmetries for (2)

And for the Lax Pair:

$$\begin{aligned} \phi_1 &= \left[-i\left(\ddot{K}_1(t)x^2 + \dot{K}_2(t)x + K_3(t)\right) - 2\dot{K}_1(t) + \lambda\right]\psi \\ \phi_2 &= \left[iC_2(y) - C_1'(y) + \lambda\right]\chi + \left[C_4(y) + iC_5(y)\right]\rho \\ \phi_3 &= \left[iC_3(y) - C_1'(y) + \lambda\right]\rho - \left[C_4(y) - iC_5(y)\right]\chi \end{aligned} \tag{6}$$

Analysis of the symmetries

- These Lie symmetries depend on a set of nine arbitrary functions:

- Three arbitrary real functions of t, $K_i(t), i = 1, ..., 3$.
- Five arbitrary real functions of y, $C_j(y), j = 1, ..., 5$.
- One arbitrary real function $\delta(y, t)$.

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- The only additional symmetry that corresponds strictly to the Lax pair itself is the one associated with λ.

Infinitesimal generators I

The ten infinitesimal generator associated to these symmetries are

$$X^{[1]}_{\{K_1(t)\}} = \frac{1}{6}x^3\dddot{K}_1\frac{\partial}{\partial m} + ix^2\ddot{K}_1\left(\alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2} - \psi\frac{\partial}{\partial\psi}\right) + 2\dot{K}_1\left(2x\frac{\partial}{\partial x} - \alpha_1\frac{\partial}{\partial\alpha_1} - \alpha_2\frac{\partial}{\partial\alpha_2} - 2m\frac{\partial}{\partial m} - \psi\frac{\partial}{\partial\psi}\right) + 8K_1\frac{\partial}{\partial t}$$

$$X^{[2]}_{\{K_2(t)\}} = \frac{1}{4}x^2\ddot{K}_2\frac{\partial}{\partial m} + ix\dot{K}_2\left(\alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2} - \psi\frac{\partial}{\partial\psi}\right) + 2K_2\frac{\partial}{\partial x}$$

$$X^{[3]}_{\{K_3(t)\}} = \frac{1}{2}x\dot{K}_3\frac{\partial}{\partial m} + iK_3\left(\alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2} - \psi\frac{\partial}{\partial\psi}\right)$$

$$Z_{\{\delta(y,t)\}} = \delta\,\frac{\partial}{\partial m}$$

Infinitesimal generators II

$$Y^{[1]}_{\{C_1(y)\}} = -C_1'\left(\alpha_1\frac{\partial}{\partial\alpha_1} + \alpha_2\frac{\partial}{\partial\alpha_2} + \chi\frac{\partial}{\partial\chi} + \rho\frac{\partial}{\partial\rho}\right) + 2C_1\frac{\partial}{\partial y}$$

$$Y^{[2]}_{\{C_2(y)\}} = iC_2\left(\alpha_1\frac{\partial}{\partial\alpha_1} + \chi\frac{\partial}{\partial\chi}\right)$$

$$Y^{[3]}_{\{C_3(y)\}} = iC_3\left(\alpha_2\frac{\partial}{\partial\alpha_2} + \rho\frac{\partial}{\partial\rho}\right)$$

$$Y^{[4]}_{\{C_4(y)\}} = C_4\left(\alpha_2\frac{\partial}{\partial\alpha_1} - \alpha_1\frac{\partial}{\partial\alpha_2} + \rho\frac{\partial}{\partial\chi} - \chi\frac{\partial}{\partial\rho}\right)$$

$$Y^{[5]}_{\{C_5(y)\}} = iC_5\left(\alpha_2\frac{\partial}{\partial\alpha_1} + \alpha_1\frac{\partial}{\partial\alpha_2} + \rho\frac{\partial}{\partial\chi} + \chi\frac{\partial}{\partial\rho}\right)$$

$$\Lambda_{\{\lambda\}} = \left(\psi\frac{\partial}{\partial\psi} + \chi\frac{\partial}{\partial\chi} + \rho\frac{\partial}{\partial\rho}\right)$$

Commutation relations

- These generators can be classified in the one associated to the arbitrary constant and the ones associated to the arbitrary functions.

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	$X^{[1]}_{\{K_1\}}$	$X^{[2]}_{\{K_2\}}$	$X^{[3]}_{\{K_3\}}$	$Z_{\{\delta\}}$
$X^{[1]}_{\{H_1\}}$	$X^{[1]}_{\{8H_1\dot{K}_1-8K_1\dot{H}_1\}}$	$X^{[2]}_{\{8H_1\dot{K}_2-4K_2\dot{H}_1\}}$	$X^{[3]}_{\{8H_1\dot{K}_3\}}$	$Z_{\{8H_1\partial_t(\delta)+4\delta\dot{H}_1\}}$
$X^{[2]}_{\{H_2\}}$	$-X^{[2]}_{\{8K_1\dot{H}_2-4H_2\dot{K}_1\}}$	$X^{[3]}_{\{2H_2\dot{K}_2-2K_2\dot{H}_2\}}$	$Z_{\{H_2\dot{K}_3\}}$	0
$X^{[3]}_{\{H_3\}}$	$-X^{[3]}_{\{8K_1\dot{H}_3\}}$	$-Z_{\{K_2\dot{H}_3\}}$	0	0
$Z_{\{\gamma\}}$	$-Z_{\{8K_1\partial_t(\gamma)+4\gamma\dot{K}_1\}}$	0	0	0

Commutation relations

	$Z_{\{\delta\}}$	$Y^{[1]}_{\{C_1\}}$	$Y^{[2]}_{\{C_2\}}$	$Y^{[3]}_{\{C_3\}}$	$Y^{[4]}_{\{C_4\}}$	$Y^{[5]}_{\{C_5\}}$
$Z_{\{\gamma\}}$	0	$-Z_{\{2\,C_1\partial_y(\gamma)\}}$	0	0	0	0
$Y^{[1]}_{\{J_1\}}$	$Z_{\{2\,J_1\partial_y(\delta)\}}$	$Y^{[1]}_{\{2(J_1\,C_1'-C_1\,J_1')\}}$	$Y^{[2]}_{\{2J_1\,C_2'\}}$	$Y^{[3]}_{\{2J_1\,C_3'\}}$	$Y^{[4]}_{\{2J_1\,C_4'\}}$	$Y^{[5]}_{\{2J_1\,C_5'\}}$
$Y^{[2]}_{\{J_2\}}$	0	$-Y^{[2]}_{\{2C_1J_2'\}}$	0	0	$-Y^{[5]}_{\{J_2\,C_4\}}$	$Y^{[4]}_{\{J_2\,C_5\}}$
$Y^{[3]}_{\{J_3\}}$	0	$-Y^{[3]}_{\{2C_1J_3'\}}$	0	0	$Y^{[5]}_{\{J_3\,C_4\}}$	$-Y^{[4]}_{\{J_3\,C_5\}}$
$Y^{[4]}_{\{J_4\}}$	0	$-Y^{[4]}_{\{2C_1J_4'\}}$	$Y^{[5]}_{\{C_2\,J_4\}}$	$-Y^{[5]}_{\{C_3\,J_4\}}$	0	$-Y^{[2]}_{\{2\,J_4\,C_5\}}+Y^{[3]}_{\{2\,J_4\,C_5\}}$
$Y^{[5]}_{\{J_5\}}$	0	$-Y^{[5]}_{\{2C_1J_5'\}}$	$-Y^{[4]}_{\{C_2\,J_5\}}$	$Y^{[4]}_{\{C_3\,J_5\}}$	$Y^{[2]}_{\{2\,C_4J_5\}}-Y^{[3]}_{\{2\,C_4J_5\}}$	0

Commutation relations

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$Z_{\{\gamma\}}$	0	$-Z_{\{2\,C_1\partial_y(\gamma)\}}$	0	0	0	0
$Y^{[1]}_{\{J_1\}}$	$Z_{\{2\,J_1\partial_y(\delta)\}}$	$Y^{[1]}_{\{2(J_1\,C_1'-C_1\,J_1')\}}$	$Y^{[2]}_{\{2J_1\,C_2'\}}$	$Y^{[3]}_{\{2J_1\,C_3'\}}$	$Y^{[4]}_{\{2J_1\,C_4'\}}$	$Y^{[5]}_{\{2J_1\,C_5'\}}$
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- $\Lambda_{\{\lambda\}}$ commutes with all other generators.
- $\left[X^{[j]}_{\{K_j(t)\}}, Y^{[l]}_{\{C_l(y)\}}\right]=0,\ j=1,\dots,3,\ l=1,\dots,5.$
- Every commutator provides another generator, in an unusual way due to the presence of the arbitrary functions.
- It is possible to construct a finite dimensional Lie Algebra by selecting special values for the arbitrary functions.

Similarity reductions

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Similarity reductions can be computed from the analysis of invariant solutions, obtained by solving the characteristic system

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{d\alpha_1}{\eta_1} = \frac{d\alpha_2}{\eta_2} = \frac{dm}{\eta_3} = \frac{d\psi}{\phi_1} = \frac{d\chi}{\phi_2} = \frac{d\rho}{\phi_3} \tag{7}$$

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In the following, we will deal with the invariant solutions

	Original variables	New reduced variables
Independent variables	x, y, t	p, q
Fields	$\alpha_1(x,y,t), \alpha_2(x,y,t), m(x,y,t)$	$F(p,q), H(p,q), N(p,q)$
Eigenfunctions	$\psi(x,y,t), \chi(x,y,t), \rho(x,y,t)$	$\Phi(p,q), \Sigma(p,q), \Omega(p,q),$

Similarity reductions

The following nontrivial cases have been considered:

- $K_1(t) \neq 0,\ K_2(t) \neq 0,\ C_1(y) \neq 0$
- $K_1(t) \neq 0,\ K_2(t) \neq 0,\ C_1(y) = 0$
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We introduce the shorthand notation:

$$I_0(t) = \frac{1}{4}\int \frac{K_2(t)}{K_1(t)^{\frac{3}{2}}}\,dt, \quad I_1(t) = \frac{1}{4}\int \frac{K_2(t)^2}{K_1(t)^2}\,dt,$$

$$I_2(t) = \frac{1}{512}\int \frac{K_2(t)^3}{K_1(t)^{\frac{5}{2}}}\,dt$$

Reduced variables

$$p = \frac{x}{\sqrt{K_1(t)}} - l_0(t), \qquad q = 4\int \frac{dy}{C_1(y)} - \int \frac{dt}{K_1(t)}$$

Reduced variables

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Reduced fields

$$\alpha_1(x,y,t) = \frac{2\,F(p,q)}{K_1(t)^{\frac{1}{4}}\ C_1(y)^{\frac{1}{2}}}\ e^{\frac{i}{8}\left(\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - I_1(t)\right)}$$

$$\alpha_2(x,y,t) = \frac{2\,H(p,q)}{K_1(t)^{\frac{1}{4}}\ C_1(y)^{\frac{1}{2}}}\ e^{\frac{i}{8}\left(\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - I_1(t)\right)}$$

$$m(x,y,t) = \frac{x^3}{24}K_1(t)^{-\frac{1}{2}}\left(K_1(t)^{\frac{1}{2}}\right)_{tt} + \frac{x^2}{32}K_1(t)^{-\frac{1}{2}}\left(\frac{K_2(t)}{\sqrt{K_1(t)}}\right)_t - \frac{1}{32}x\dot{I}_1(t) + \frac{N(p,q)+I_2(t)}{\sqrt{K_1(t)}}$$

I. $K_1(t) \neq 0,\ K_2(t) \neq 0,\ C_1(y) \neq 0$

Reduced eigenfunctions

$$\psi(x,y,t) = \frac{\Phi(p,q)}{2\,K_1(t)^{\frac{1}{4}}}\, e^{-\frac{i}{8}\left(\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - l_1(t)\right) + \frac{\lambda}{8}\int \frac{dt}{K_1(t)}}$$

$$\chi(x,y,t) = \frac{\Sigma(p,q)}{C_1(y)^{\frac{1}{2}}}\, e^{\frac{\lambda}{8}\int \frac{dt}{K_1(t)}}, \qquad \rho(x,y,t) = \frac{\Omega(p,q)}{C_1(y)^{\frac{1}{2}}}\, e^{\frac{\lambda}{8}\int \frac{dt}{K_1(t)}}$$

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Reduced equations

$$iF_q - F_{pp} - 2FN_p = 0$$
$$iH_q - H_{pp} - 2HN_p = 0$$
$$\left(N_q + FF^{\dagger} + HH^{\dagger}\right)_p = 0$$

which is a nonlocal multi-component NLS Eq. in $1+1$ dim. for $\{F^{\dagger}, H^{\dagger}\}$ and density N_q.

I. $K_1(t) \neq 0$, $K_2(t) \neq 0$, $C_1(y) \neq 0$

Reduced spectral problem

- p-Lax Pair

$$\Phi_{pp} + \left(2N_p - \frac{i}{8}\lambda\right)\Phi - iF^\dagger\Sigma - iH^\dagger\Omega = 0$$
$$\Sigma_p + F\,\Phi = 0$$
$$\Omega_p + H\,\Phi = 0$$

- q-Lax Pair

$$\Phi_q + F^\dagger\Sigma + H^\dagger\Omega = 0$$
$$\Sigma_q + i\,(F\,\Phi_p - F_p\,\Phi) - \frac{\lambda}{8}\Sigma = 0$$
$$\Omega_q + i\,(H\,\Phi_p - H_p\,\Phi) - \frac{\lambda}{8}\Omega = 0$$

- λ plays the role of the spectral parameter.

II. $K_1(t) \neq 0,\ K_2(t) \neq 0,\ C_1(y) = 0$

Reduced variables

$$p = \frac{x}{\sqrt{K_1(t)}} - l_0(t), \qquad q = y$$

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$$p = \frac{x}{\sqrt{K_1(t)}} - l_0(t), \qquad q = y$$

Reduced fields

$$\alpha_1(x,y,t) = \frac{F(p,q)}{K_1(t)^{\frac{1}{4}}}\, e^{\frac{i}{8}\left(\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - l_1(t)\right)}$$

$$\alpha_2(x,y,t) = \frac{H(p,q)}{K_1(t)^{\frac{1}{4}}}\, e^{\frac{i}{8}\left(\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - l_1(t)\right)}$$

$$m(x,y,t) = \frac{x^3}{24}K_1(t)^{-\frac{1}{2}}\left(K_1(t)^{\frac{1}{2}}\right)_{tt} + \frac{x^2}{32}K_1(t)^{-\frac{1}{2}}\left(\frac{K_2(t)}{\sqrt{K_1(t)}}\right)_t - \frac{1}{32}x\dot{l}_1(t) + \frac{N(p,q)+l_2(t)}{\sqrt{K_1(t)}}$$

II. $K_1(t) \neq 0,\ K_2(t) \neq 0,\ C_1(y) = 0$

Reduced eigenfunctions

$$\psi(x,y,t) = \frac{\Phi(p,q)}{K_1(t)^{\frac{1}{4}}}\, e^{-\frac{i}{8}\left(\frac{\dot{K}_1(t)}{K_1(t)}x^2 + \frac{K_2(t)}{K_1(t)}x - I_1(t)\right) + \frac{\lambda}{8}\int \frac{dt}{K_1(t)}}$$

$$\chi(x,y,t) = \Sigma(p,q)\, e^{\frac{\lambda}{8}\int \frac{dt}{K_1(t)}}, \qquad \rho(x,y,t) = \Omega(p,q)\, e^{\frac{\lambda}{8}\int \frac{dt}{K_1(t)}}$$

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Reduced equations

$$F_{pp} + 2FN_p = 0$$

$$H_{pp} + 2HN_p = 0$$

$$\left(N_q + FF^{\dagger} + HH^{\dagger}\right)_p = 0$$

II. $K_1(t) \neq 0,\ K_2(t) \neq 0,\ C_1(y) = 0$

Reduced spectral problem

- p-Lax Pair

$$\Phi_{pp} + (2N_p - \frac{i}{8}\lambda)\Phi = 0$$

$$\Sigma_p + F\,\Phi = 0, \qquad \Omega_p + H\,\Phi = 0$$

- q-Lax Pair

$$\Phi_q + F^\dagger\,\Sigma + H^\dagger\,\Omega = 0$$

$$\lambda\Sigma - 8i\,(F\,\Phi_p - F_p\,\Phi) = 0, \qquad \lambda\Omega - 8i\,(H\,\Phi_p - H_p\,\Phi) = 0$$

- Or equivalently, the scalar Lax pair in $1+1$

$$\Phi_{pp} + (2N_p - \frac{i}{8}\lambda)\Phi = 0$$

$$\lambda\Phi_q - 8i\left\{\left(F^\dagger\,F_p + H^\dagger\,H_p\right)\,\Phi + N_q\,\Phi_p\right\} = 0$$

III. $K_2(t) \neq 0,\ C_1(y) \neq 0,\ K_1(t) = 0$

Reduced variables

$$p = \frac{x}{K_2(t)} - \int \frac{dy}{C_1(y)}, \qquad q = \int \frac{dt}{K_2(t)^2}$$

III. $K_2(t) \neq 0,\ C_1(y) \neq 0,\ K_1(t) = 0$

Reduced variables

$$p = \frac{x}{K_2(t)} - \int \frac{dy}{C_1(y)}, \qquad q = \int \frac{dt}{K_2(t)^2}$$

Reduced fields

$$\alpha_1(x,y,t) = \frac{F(p,q)}{K_2(t)^{\frac{1}{2}}\, C_1(y)^{\frac{1}{2}}}\, e^{\frac{i}{4}\left(\frac{\dot{K}_2(t)}{K_2(t)}x^2 + 2p - q\right)}$$

$$\alpha_2(x,y,t) = \frac{H(p,q)}{K_2(t)^{\frac{1}{2}}\, C_1(y)^{\frac{1}{2}}}\, e^{\frac{i}{4}\left(\frac{\dot{K}_2(t)}{K_2(t)}x^2 + 2p - q\right)}$$

$$m(x,y,t) = \frac{1}{24}\frac{\ddot{K}_2(t)}{K_2(t)}x^3 + \frac{N(p,q)}{K_2(t)}$$

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Reduced eigenfunctions

$$\psi(x,y,t) = \frac{\Phi(p,q)}{K_2(t)^{\frac{1}{2}}}\, e^{-\frac{i}{4}\left(\frac{\dot{K}_2(t)}{K_2(t)}x^2 - q\right) + \frac{\lambda}{2}\int \frac{dy}{C_1(y)}}$$

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Reduced equations

$$iF_q + (F_p + iF)_p + 2FN_p = 0$$

$$iH_q + (H_p + iH)_p + 2HN_p = 0$$

$$\left(N_p - FF^{\dagger} - HH^{\dagger}\right)_p = 0$$

III. $K_2(t) \neq 0,\ C_1(y) \neq 0,\ K_1(t) = 0$

Reduced spectral problem

- p-Lax Pair

$$\Phi_p - \left(F^\dagger \Sigma + H^\dagger \Omega\right) - \frac{\lambda}{2}\Phi = 0$$

$$\Sigma_p + F\,\Phi + \frac{i}{2}\Sigma = 0, \qquad \Omega_p + H\,\Phi + \frac{i}{2}\Omega = 0$$

- q-Lax Pair

$$\Phi_q - i\left(FF^\dagger + HH^\dagger - 2N_p - \frac{(\lambda^2+1)}{4}\right)\Phi + \left(iF_p^\dagger + \frac{i\lambda+1}{2}F^\dagger\right)\Sigma + \left(iH_p^\dagger + \frac{i\lambda+1}{2}H^\dagger\right)\Omega = 0$$

$$\Sigma_q - \left(\frac{(i\lambda+1)}{2}F - iF_p\right)\Phi - iFF^\dagger\Sigma - iFH^\dagger\Omega = 0$$

$$\Omega_q - \left(\frac{(i\lambda+1)}{2}H - iH_p\right)\Phi - iHF^\dagger\Sigma - iHH^\dagger\Omega = 0$$

Conclusions and future perspectives

- Classical Lie symmetries have been determined for a generalized multi-component NLS Eq. in $2+1$ dimensions (1) and its Lax Pair, in terms of nine arbitrary functions and a single arbitrary constant.

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- Remark that three special reductions lead to nontrivial problems in $1+1$ dimensions.
- The spectral parameter arises naturally in the process of constructing the reductions, due to the symmetry associated to the arbitrary constant.

References

- P. Albares, J. M. Conde, P. G. Estévez (2018), arXiv:1807.09039v1 [nlin.SI].
- P. Albares *et al.* (2018), *Phys. Rev. E*, 97, 022210.
- E. Díaz *et al.* (2018), *New J. Phys.*, 20, 043055.
- M. Legaré (1996), *J. Nonlinear Math. Phys.*, 3, 266–285.
- X. Lu, M. Peng (2013), *Nonlinear Dyn.*, 73, 405.
- S. V. Manakov (1974), *Sov. Phys. JETP*, 38, no. 2, 248–253
- P. J. Olver (1993). *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York.
- H. Stephani (1989). *Differential Equations. Their solutions using symmetries*, edited by M.Mac Callum, Cambridge Univ. Press.
- D. S. Wang, D .J. Zhang, J. Yang (2010), *J. Math. Phys.*. 51, 023510.

Thank you for your attention!