INSTITUTO DE CIENCIAS MATEMÁTICAS

UNIVERSIDAD AUTONOMA DE MADRID

Groupoids and Distributions:

A new way to deal with non-uniform material bodies

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Joint work with M. de León and M. Epstein

- Groupoids and Constitutive theory: A (very) brief introduction.

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- Generalized homogeneity.

Groupoids and Constitutive theory: A (very) brief introduction

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- *Response functional*: $W : Gl(3, \mathbb{R}) \times \mathcal{B} \to V$

1-jets Lie groupoid on $\mathcal{B}$

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$$W : \Pi^1(\mathcal{B},\mathcal{B}) \to V$$

Definition 1.1

A body $\mathcal{B}$ is said to be *uniform* if for each two points $X, Y \in \mathcal{B}$ there exists a local diffeomorphism ψ from an open neighbourhood $\mathcal{U} \subseteq \mathcal{B}$ of X to an open neighbourhood $\mathcal{V} \subseteq \mathcal{B}$ of Y such that $\psi(X) = Y$ and

$$W\left(j^1_{Y,\kappa(Y)}\kappa \cdot j^1_{X,Y}\psi\right) = W\left(j^1_{Y,\kappa(Y)}\kappa\right), \tag{1}$$

for all infinitesimal deformation $j^1_{Y,\kappa(Y)}\kappa$. Any $1-$jet $j^1_{X,Y}\psi$ satisfying Eq. (1) is called *material isomorphism*.

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Material groupoid: $\Omega(\mathcal{B}) \rightrightarrows \mathcal{B}$

Definition 1.2

A body $\mathcal{B}$ is said to be *smoothly uniform* if for each point $X \in \mathcal{B}$ there is an infinitesimal neighbourhood $\mathcal{U}$ around X such that for all $Y \in \mathcal{U}$ there exists a smooth field of material isomorphisms P from $\epsilon(X)$ to a material isomorphism $j^1_{Y,X}\phi$.

Definiton 1.3

A simple body $\mathcal{B}$ is said to be *homogeneous* if it admits a (global) configuration ψ of $\mathcal{B}$ which induces a smooth field of material isomorphisms P, i.e.,

$$P(X,Y) = j^1_{X,Y}\left(\psi^{-1} \circ \tau_{\psi(Y)-\psi(X)} \circ \psi\right), \ \forall X,Y \in \mathcal{B},$$

where $\tau_{\psi(Y)-\psi(X)} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\psi(Y) - \psi(X)$. $\mathcal{B}$ is said to be *locally homogeneous* if there exists a covering of $\mathcal{B}$ by homogeneous open sets.

Proposition 1.1

Let $\mathcal{B}$ be a body. $\mathcal{B}$ is uniform if, and only if, $\Omega(\mathcal{B})$ is a transitive subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.

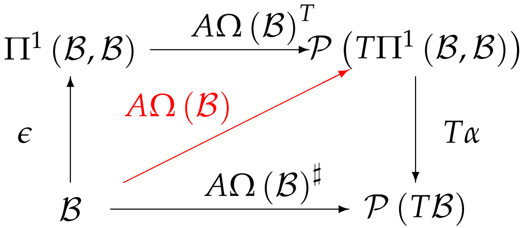
Proposition 1.2

Let $\mathcal{B}$ be a body. $\mathcal{B}$ is smoothly uniform if, and only if, $\Omega(\mathcal{B})$ is a transitive Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.

Proposition 1.3

Let $\mathcal{B}$ be a body. $\mathcal{B}$ is homogeneous if, and only if, $\Omega(\mathcal{B})$ is an *integrable* Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.

Material distribution


$\Pi^1(\mathcal{B},\mathcal{B})$
$A\Omega(\mathcal{B})^T$
$\mathcal{P}(T\Pi^1(\mathcal{B},\mathcal{B}))$
$A\Omega(\mathcal{B})$
ϵ
$T\alpha$
$\mathcal{B}$
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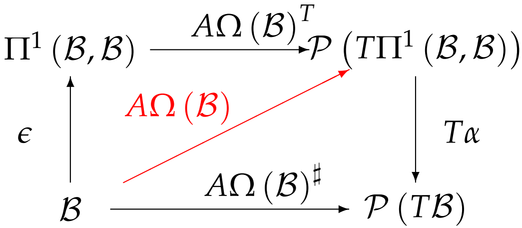
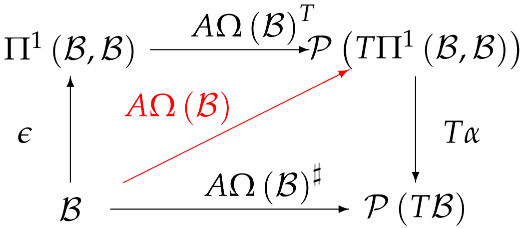


$$\overline{\mathcal{H}}\left(g\right) \mapsto \Omega\left(\mathcal{B}\right)$$

$$\mathcal{H}\left(X\right) \mapsto \mathcal{B}$$

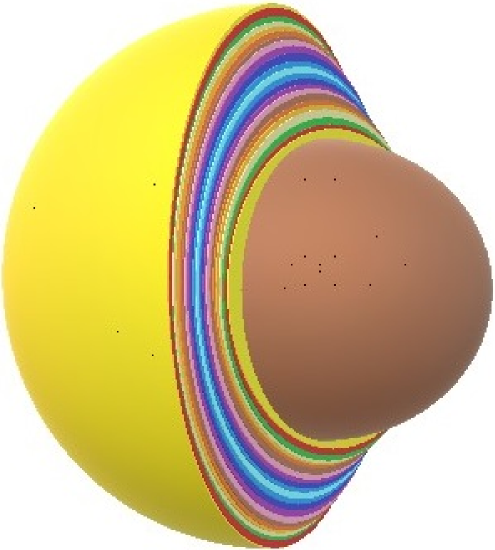


$$\overline{\mathcal{H}}(g) \mapsto \Omega(\mathcal{B})$$

$$\mathcal{H}(X) \mapsto \mathcal{B}$$

Theorem 2.1

For all $X \in \mathcal{B}$, $\Omega(\mathcal{H}(X))$ is a transitive Lie subgroupoid of $\Pi^1(\mathcal{B},\mathcal{B})$. Thus, any body $\mathcal{B}$ can be covered by a foliation of smoothly uniform material submanifolds.



Graded uniformity

Definition 3.1

Let $\mathcal{B}$ be a simple body. $\mathcal{B}$ is said to be *uniform of grade p at $X \in \mathcal{B}$* if $A\Omega\left(\mathcal{B}\right)^{\sharp}_{X}$ has dimension p. $\mathcal{B}$ is *uniform of grade p* if it is uniform of grade p at all its points.

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Smoothly uniform $\Leftrightarrow$ Uniform of grade 3

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Smoothly uniform $\Leftrightarrow$ Uniform of grade 3

Laminated body $\Leftrightarrow$ Uniform of grade 2

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Smoothly uniform $\Leftrightarrow$ Uniform of grade 3

Laminated body $\Leftrightarrow$ Uniform of grade 2

Filamented bundles $\Leftrightarrow$ Uniform of grade 1

Corollary 3.1

Let $\mathcal{B}$ be a body and let $X \in \mathcal{B}$. $\mathcal{B}$ is uniform of grade p at X if, and only if, the uniform body submanifold at X (that is, the leaf $\mathcal{H}(X)$ of the body foliation) has dimension p.

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Corollary 3.2

Let $\mathcal{B}$ be a body. $\mathcal{B}$ is uniform of grade p if, and only if, the body foliation $\mathcal{H}$ is regular of rank p.

Corollary 3.3

Let $\mathcal{B}$ be a body and let $X \in \mathcal{B}$. $\mathcal{B}$ is uniform of grade greater or equal to p at X if, and only if, there exists a foliation $\{\mathcal{H}'(X)\}$ by smoothly uniform submanifolds of $\mathcal{B}$ such that the leaf $\mathcal{H}'(X)$ has dimension greater or equal to p.

Corollary 3.3

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Corollary 3.4

Let $\mathcal{B}$ be a body. $\mathcal{B}$ is uniform of grade p or greater if, and only if, the body can be foliated by smoothly uniform material submanifolds of dimension p.

Generalized homogeneity

Defintion 4.1

Let $\mathcal{B}$ be a simple body and $\mathcal{N}$ be a submanifold of $\mathcal{B}$. $\mathcal{N}$ is said to be *locally homogeneous* if, and only if, for all point $X \in \mathcal{N}$ there exists a local configuration $\kappa_{\mathcal{U}}$ of $\mathcal{B}$, with $X \in \mathcal{U}$, which satisfies that

$$j^1_{Y,Z}\left(\kappa_{\mathcal{U}}^{-1} \circ \tau_{\kappa_{\mathcal{U}}(Z)-\kappa_{\mathcal{U}}(Y)} \circ \kappa_{\mathcal{U}}\right),$$

is a material isomorphism for all $Y, Z \in \mathcal{U} \cap \mathcal{N}$. We will say that $\mathcal{N}$ is *homogeneous* if $\mathcal{U}$ cover $\mathcal{S}$ (i.e., $\mathcal{N} \subseteq \mathcal{U}$).

Definition 4.2

Let $\mathcal{B}$ be a simple body. $\mathcal{B}$ is said to be *locally homogeneous* if, and only if, for all point $X \in \mathcal{B}$ there exists a local configuration ψ of $\mathcal{B}$, with $X \in U$, which is a foliated chart and it satisfies that

$$j^1_{Y,Z}\left(\psi^{-1} \circ \tau_{\psi(Z)-\psi(Y)} \circ \psi\right),$$

is a material isomorphism for all $Z \in U \cap \mathcal{H}(Y)$. We will say that $\mathcal{B}$ is *homogeneous* if $U = \mathcal{B}$.

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- **A smoothly uniform body $\mathcal{B}$ is homogeneous** if, and only if, $\mathcal{B}$ is homogeneous "in the usual sense".
- **A laminated (bundle) body $\mathcal{B}$ is strongly homogeneous** if, and only if, $\mathcal{B}$ is homogeneous and the homogeneous charts are also relaxable.

Proposition 4.2

Let $\mathcal{B}$ be a simple body. $\mathcal{B}$ is homogeneous if, and only if, for each $X \in \mathcal{B}$ there exists a local chart $\left(X^I\right)$ on X such that,

$$\frac{\partial W}{\partial X^L} = 0,$$

for all $L \leq dim\left(\mathcal{H}\left(X\right)\right)$.

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- **V. M. Jiménez, M. de León and M. Esptein**, *Characteristic distribution: An application to material bodies*, Journal of Geometry and Physics, 127 (2018) 19-31.
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MUITO OBRIGADO!