From (quantum) geometry to spectral theory

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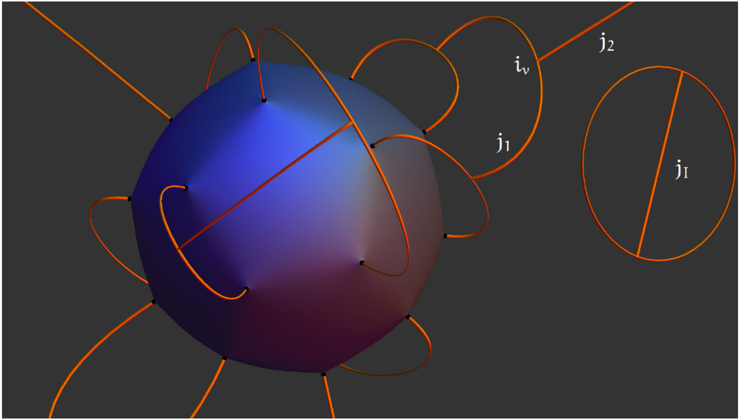
Introduction

Goal

To understand some features of the spectrum of the area operator



Eigenvalues of the (simplified) area operator

$$a_s = \sum_{n_i \in M_N} \sqrt{n_i(n_i+2)}$$

$$\left|\begin{array}{l} M_N := \{n_1, \ldots, n_N\} \in (\mathbb{N}^+)^N \\ N \in \mathbb{N} \end{array}\right.$$

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$$M_N = \{\underbrace{1, \ldots, 1}_{N_1}, \underbrace{2, \ldots, 2}_{N_2}, \underbrace{3, \ldots, 3}_{N_3}, \ldots\} \equiv (N_1, N_2, \ldots) \in c_{00}(\mathbb{N})$$

Eigenvalues of the (simplified) area operator

eigenvalue	$(N_1, N_2, \ldots)$
0	$(0,0,0,0\ldots)$
$\sqrt{3}$	$(1,0,0,0\ldots)$
$2\sqrt{2}$	$(0,1,0,0\ldots)$
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$\sqrt{15}$	$(0,0,1,0\ldots)$
$2\sqrt{2}+\sqrt{3}$	$(1,1,0,0\ldots)$
$2\sqrt{6}$	$(0,0,0,1\ldots)$
$3\sqrt{3}$	$(3,0,0,0\ldots)$
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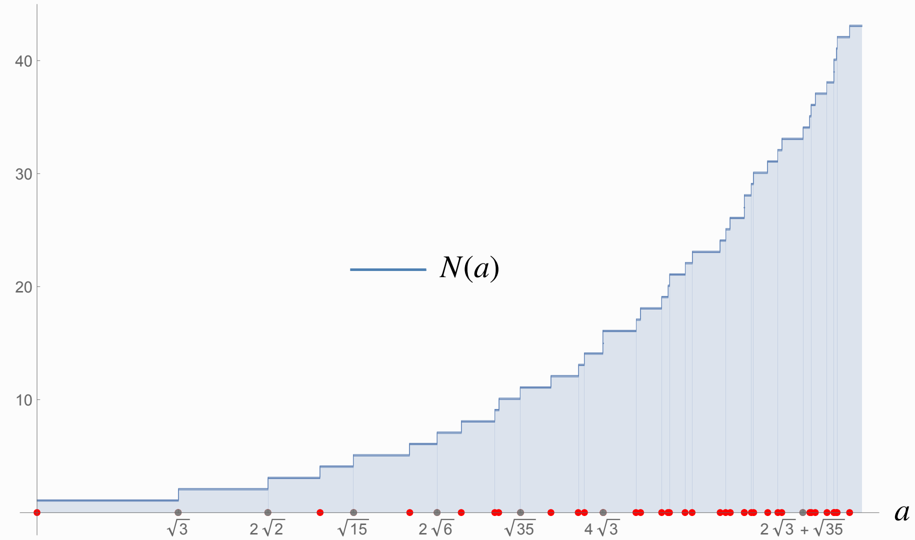

First approach: approximations

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$$\sqrt{x(x+2)} \sim x+1-\frac{1}{2x}+O\left(\frac{1}{x^2}\right) \quad \longrightarrow \quad \sqrt{n_i(n_i+2)} \sim n_i+1$$

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$$N_-(a) \leq N(a) \leq N_+(a)$$

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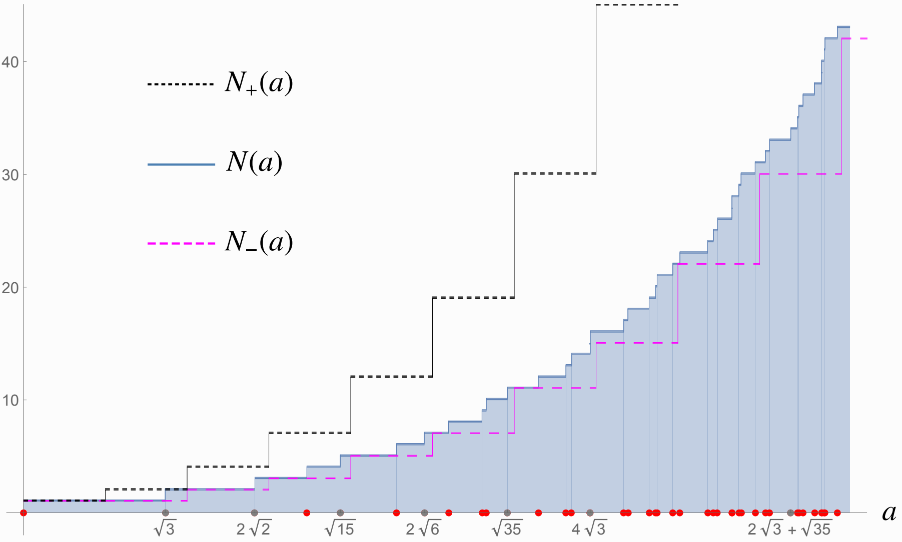

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Playing with partitions

Using generating functions and ideas of Hardy, Ramanujan, Littlewood, Rademacher. . . we obtain

$$N_-(a) \sim \frac{1}{4a\sqrt{3}}\mathrm{Exp}\left(\pi\sqrt{\frac{2a}{3}}\right) \qquad N_+(a) \sim \frac{1}{2\pi\sqrt{2a}}\mathrm{Exp}\left(\pi\sqrt{\frac{2a}{3}}\right)$$

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- Same exponent factor but different prefactor.
- We can use $n_i + \varepsilon < \sqrt{n_i(n_i+2)} < n_i + 1$ for $\varepsilon \in (0,1)$.
- No clear approximation for N. New ideas are needed!

Second approach: Laplace transform

N as a staircase function

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$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{za}}{z}\mathrm{Exp}\left[-\sum_{k=1}^{\infty} \ln\left(1 - e^{-zs_k}\right)\right]\mathrm{d}z - \theta(a)$$

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- $s_k = \sqrt{k(k+2)}$ our problem at hand

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$$N(a) \sim \frac{1}{2\pi^{3/2} i}\int_{c-i\infty}^{c+i\infty} z^{1/2}\mathrm{Exp}\left((a-a_*)z + \frac{\pi^2}{6z} + \frac{z}{4}\ln z\right)F(z)\mathrm{d}z$$

Approximating N

$$N(a) \sim \frac{1}{\sqrt{\pi}} \alpha^{3/2} e^{-\alpha/2} F(\alpha)\, I_{-3/2}\left(\frac{\pi^2}{3\alpha} + \frac{\alpha}{4}\right) =: N_0(a)$$

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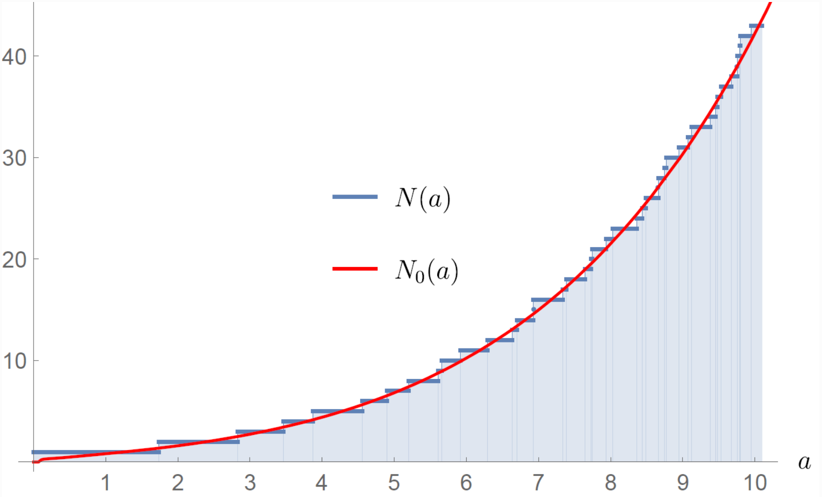
How good are our results?



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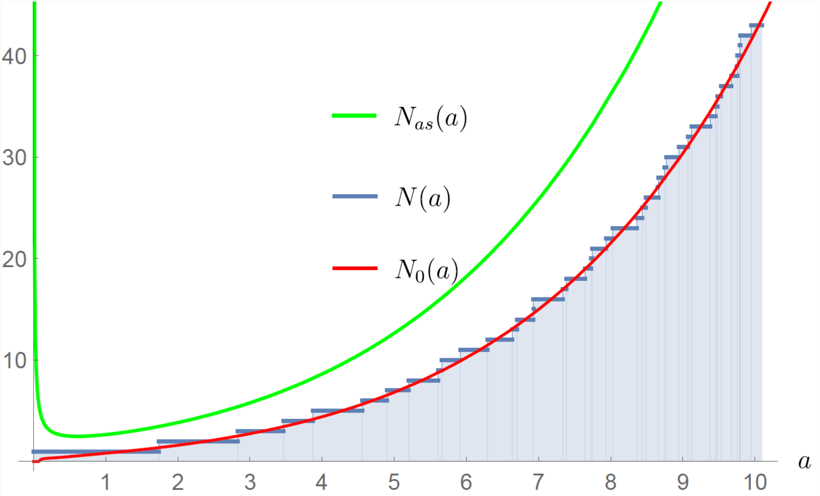

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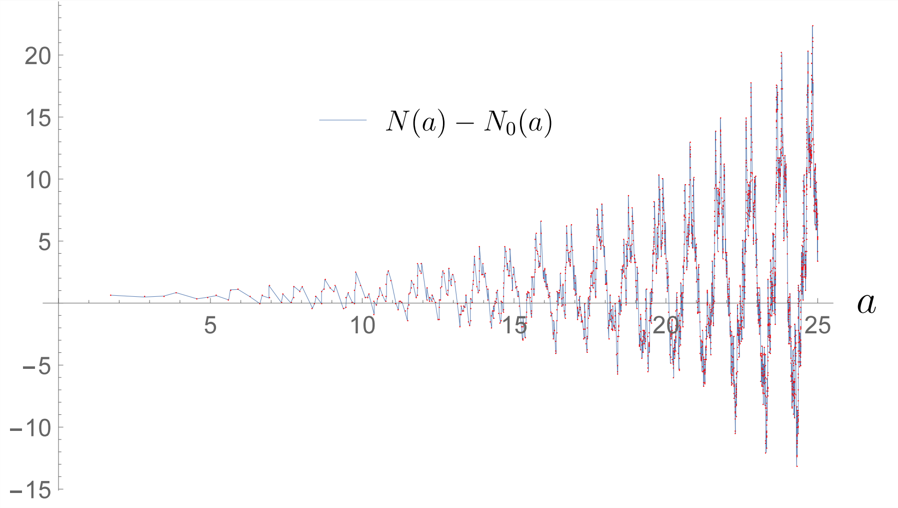

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- $N_{-}(a)$ is really good, it only fails by a **constant** $\sqrt{2}$ factor. Difficult to guess *a priori*.
- We get a very accurate representation for $N(a)$ valid for all areas.
- The used methods may be useful to tackle similar problems.

Thanks for your attention

- J. Fernando Barbero, Juan Margalef-Bentabol, Eduardo J.S. Villaseñor, *On the distribution of the eigenvalues of the area operator in loop quantum gravity*. CQG 35 (2018), [arXiv:1712.06918]
- M. Barreira, M. Carfora, C. Rovelli, *Physics with nonperturvbative quantum gravity: Radiation from a quantum black hole*. GRG 28 (1996).
- A. Ashtekar, J. Lewandowski, *Quantum Theory of Gravity I: Area Operators*. CQG 14 (1997).
- I. Agullo, F. Barbero, E.F. Borja, J. Diaz-Polo, E.J.S. Villaseñor, *Detailed black hole state counting in loop quantum gravity*. PRD 82 (2010)

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N as a staircase function

a	$N_{as}(a)$	$N_0(a)$	$N(a)$
10	68	42	43
30	8599	6282	6282
50	307719	237955	237788
70	6099037	4870521	4867770