Necessary Condition for Local Controllability

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Introduction

- Local controllability of nonlinear affine systems

Outline

- Two definitions for local controllability
- Known result for scalar-input systems
- New result for systems with two controls
- Example of application: magnetic microswimmer

Introduction

Let f_0, f_1 be analytic vector fields on $\mathbf{R}^n$. Consider the control-affine system with scalar-input control

$$\dot{z} = f_0(z) + u_1(t) f_1(z), \tag{1}$$

with $f_0(0) = 0$ (meaning that $(0, 0)$ is an equilibrium point for the system).

- Is this system locally controllable around 0?

Small-Time Local Controllability (STLC)

Definition 1 (STLC)

The control system (1) *is STLC at* $(0,0)$ *if, for every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that, for every* z_0, z_1 *in the ball centered at* 0 *with radius* η*, there exists a control* $u(\cdot)$ *in* $L^\infty([0,\varepsilon])$ *such that the solution of the control system* $z(\cdot) : [0,\varepsilon] \to \mathbf{R}^n$ *of* (1) *satisfies* $z(0) = z_0$*,* $z(\varepsilon) = z_1$*, and*

$$\|u\|_{L^\infty} \leqslant \varepsilon\,.$$

Definition 2 (α-STLC)

Let $\alpha \geqslant 0$*. The control system* (1) *is* α*-STLC at* $(0,0)$ *if, for every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that, for every* z_0, z_1 *in the ball centered at* 0 *with radius* η*, there exists a control* $u(\cdot)$ *in* $L^\infty([0,\varepsilon])$ *such that the solution of the control system* $z(\cdot) : [0,\varepsilon] \to \mathbf{R}^n$ *of* (1) *satisfies* $z(0) = z_0$*,* $z(\varepsilon) = z_1$*, and*

$$\|u\|_{L^\infty} \leqslant \alpha + \varepsilon\,.$$

Let us call:

- S_k the subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ spanned by all the Lie brackets of f_0, f_1 containing f_1 at most k times;
- $S_k(0)$ the subspace of $\mathbf{R}^n$ spanned by the value at 0 of the elements of S_k.

Sufficient condition for STLC:

Theorem 1 (Sussmann 1983)

If, for all $k \geqslant 1$, $S_{2k}(0)$ is included in $S_{2k-1}(0)$, then (1) is STLC.

Necessary condition for STLC:

Theorem 2 (id.)

If $[f_1, [f_0, f_1]](0) \notin S_1(0)$, then (1) is not STLC($\alpha$) for any α.

What happens with a second control ?

Consider the new system

$$\dot{z} = f_0(z) + u_1(t) f_1(z) + u_2(t) f_2(z) \qquad (2)$$

with $f_0(0) = 0$ and $f_2(0) = 0$.

Similarly to the scalar case, we call:

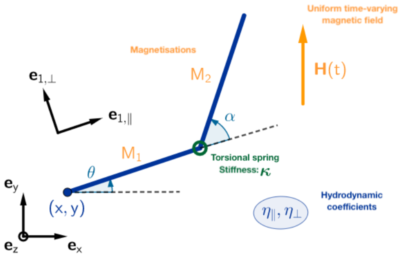
- R_1 the subspace of $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ spanned by all the Lie brackets of f_0, f_1, f_2 containing f_1 at most one time;
- $R_1(0)$ the subspace of $\mathbf{R}^n$ spanned by the value at 0 of the elements of S_1.

Theorem 3

Assume that $[f_1, [f_0, f_1]](0) \notin R_1(0)$. Then,

1. *if $[f_1, [f_0, f_1]](0) \in Span(R_1(0), [f_1, [f_2, f_1]](0))$, (2) is not STLC.*
2. *else, (2) is not STLC(α) for any α.*

Application : two-link magnetic micro-swimmer



- Hydrodynamics coefficients (Resistive Force Theory): $\eta_{\|}$, $\eta_{\perp}$
- Magnetic moments: M_1, M_2
- Uniform time-varying magnetic field: $\mathbf{H}(t) \to$ control function with two scalar components projected on the moving basis $(\mathbf{e}_{1,\|}, \mathbf{e}_{1,\perp})$

Writing balance of forces and moments give a control system :

$$\dot{\mathbf{z}} = f_0(\mathbf{z}) + H_{\perp}\, f_1(\mathbf{z}) + H_{\|}\, f_2(\mathbf{z})\,.$$

where the state is $\mathbf{z} = (x, y, \theta, \alpha)$ and $H_{\|}$ et $H_{\perp}$ the controls. Moreover, 0 is an equilibrium, and $f_0(0) = f_2(0) = 0$.

Application : two-link magnetic micro-swimmer

Suppose $M_1 \neq 0$, $M_2 \neq 0$, $M_1 + M_2 \neq 0$ and $\eta \neq \xi$. Then one can check that $R_1(0) = \text{Vect}\,(e_2, e_3, e_4)$, and

$$[f_1, [f_0, f_1]](0) = (\frac{216\, M_1 M_2\,(M_1 - M_2)}{\ell^8 \eta^2 \xi}, 0, 0, 0),$$

$$[f_1, [f_2, f_1]](0) = (-\frac{216\kappa(M_1 + M_2)(M_1 - M_2)}{\ell^8 \eta^2 \xi}, 0, 0, 0).$$

Therefore:

Theorem 4

The two-link swimmer is not STLC.

But one can also show:

Theorem 5 (Giraldi, Pomet, 2017)

The two-link swimmer is K-STLC with $K = 2\,\kappa \left| \dfrac{M_2 + M_1}{M_2\, M_1} \right|$.

Perspectives

Possible extensions for the result: second obstruction when $[f_1, [f_0, f_1]](0) \in R_1(0)$, but not $[[f_0, f_1], [f_0, [f_0, f_1]]](0)$.