Short talk: Numerical optimal control of nonholonomic mechanical systems.

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Mechanical systems

Mechanical systems

- Described via Lagrangian or Hamiltonian formulation
- Built-in geometric properties (Manifold structure of configuration space, symplecticity)
- Built-in conservation laws due to symmetries (Noether theorem)

Symplectic integrators

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- Good qualitative and quantitative behaviour.

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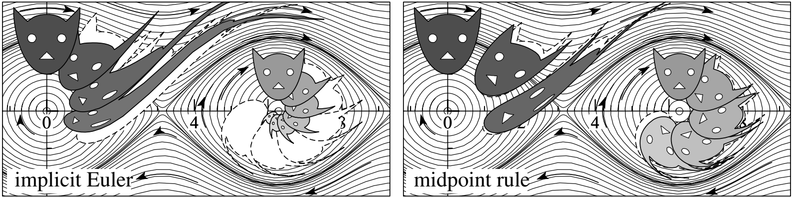

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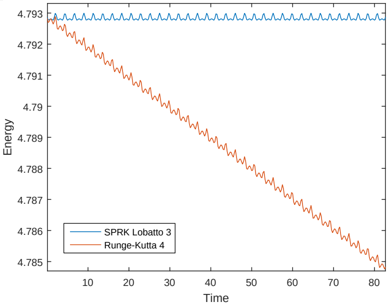

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- Preserve flow properties (symplecticity, momenta...).
- Energy not exactly preserved... [Ge & Marsden]
- ... but good long-term energy behaviour.

How come energy behaves so well?

Theorems ([Moser], [Benettin & Giorgilli], [Tang], [Murua]...) warrant that symplectic integrators are integrating exactly some existing Hamiltonian system that is *close* to the original one.

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Generating symplectic integrators easily

Variational integrators are always symplectic.

Idea ([Veselov], [Suris], [Marsden & West]...)

- Substitute continuous state space with discrete one.

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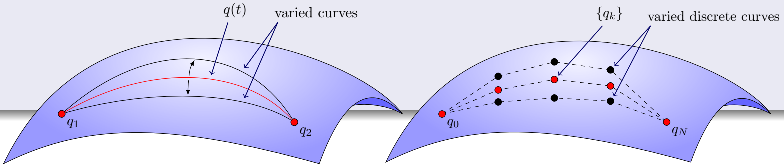

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- Derive equations of motion and conservations from the principle.

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- Substitute continuous state space with discrete one.
- Build discrete analogue of Hamilton's principle.
- Derive equations of motion and conservations from the principle.

Discrete equations of motion = Difference equations (a.k.a. **our integrator**).

The starting point

Hamilton-Pontryagin action

$(q, v, p) : [0, T] \subset \mathbb{R} \to TQ \oplus T^*Q$, sufficiently differentiable curve and fixed boundary values $q(0) = q_a$, $q(T) = q_b$.

$$\mathcal{J}_{\mathcal{HP}}(q, v, p) = \int_0^T \left[L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \right] \mathrm{d}t$$

Dynamical equations

$$\begin{aligned} \frac{\mathrm{d}q(t)}{\mathrm{d}t} &= v(t), \\ \frac{\mathrm{d}p(t)}{\mathrm{d}t} &= D_1 L(q(t), v(t)), \\ p(t) &= D_2 L(q(t), v(t)) \end{aligned}$$

Variationally partitioned Runge-Kutta integrators

Discrete Hamilton-Pontryagin action

$$
\begin{aligned}
(\mathcal{J}_{\mathcal{HP}})_d = \sum_{k=0}^{N-1} \sum_{i=1}^{s} h b_i \Bigg[& L\left(Q_k^i, V_k^i\right) + \left\langle P_k^i, \frac{Q_k^i - q_k}{h} - \sum_{j=1}^{s} a_{ij} V_k^j \right\rangle \\
& + \left\langle p_{k+1}, \frac{q_{k+1} - q_k}{h} - \sum_{j=1}^{s} b_j V_k^j \right\rangle \Bigg]
\end{aligned}
$$

where (a_{ij}, b_j) coefficients of a Runge-Kutta (RK) method.

Discrete dynamics

Discrete dynamical equations: Symplectic partitioned RK methods

$$q_{k+1} = q_k + h\sum_{j=1}^{s} b_j V_k^j, \quad D_2L(q_{k+1}, v_{k+1}) = D_2L(q_k, v_k) + h\sum_{i=1}^{s} \hat{b}_j D_1L(Q_k^j, V_k^j),$$

$$Q_k^i = q_k + h\sum_{j=1}^{s} a_{ij} V_k^j, \qquad D_2L(Q_k^i, V_k^i) = D_2L(q_k, v_k) + h\sum_{j=1}^{s} \hat{a}_{ij} D_1L(Q_k^j, V_k^j),$$

where $(\hat{a}_{ij}, \hat{b}_j)$ satisfy $b_i\hat{a}_{ij} + \hat{b}_j a_{ji} = b_i\hat{b}_j$ and $\hat{b}_i = b_i$.

Here we have used the continuous relation $p(t) = D_2L(q(t), v(t))$ to relate p_k and p_{k+1} with v_k and v_{k+1}.

Controlled mechanical systems

Controlled dynamics

Let $f : TQ \times U \to T^*Q$ define a mechanical forcing, $U \subset \mathbb{R}^k$ control space.

$$\begin{aligned} \frac{\mathrm{d}q(t)}{\mathrm{d}t} &= v(t), \\ \frac{\mathrm{d}p(t)}{\mathrm{d}t} &= D_1 L(q(t), v(t)) + f(q(t), v(t), u(t)), \\ p(t) &= D_2 L(q(t), v(t)) \end{aligned}$$

$(q, v, u) : [0, T] \subset \mathbb{R} \to TQ \times U$. Controlled eqs. of motion define section of $TT^*Q \times U$, of the form $(q, G(q, v), v, F(q, v, u), u)$.

Action principle for optimal control

Cost function

$$C : TQ \times U \to \mathbb{R}$$

Hamilton-Pontryagin action

$(q, p, \xi_q, \xi_p, \mu_q, \mu_p, u) : [0, T] \subset \mathbb{R} \to TT^*Q \oplus T^*T^*Q \times U$ sufficiently differentiable curve with fixed boundary values $(q(0), p(0)) = (q_a, p_a)$, $(q(T), p(T)) = (q_b, p_b)$.

$$\mathcal{J}_{\mathcal{HP}}(q, p, \xi_q, \xi_p, \mu_q, \mu_p, u) = \int_0^T [C(q(t), \xi_q(t), u(t)) + \langle(\mu_q(t), \mu_p(t)), (\dot{q}(t) - \xi_q(t), \dot{p}(t) - \xi_p(t))\rangle] \, \mathrm{d}t$$

evaluated over the mechanical section $(q, G(q, v), v, F(q, v, u), \mu_q, \mu_p, u)$.

Optimal control

Necessary conditions for optimality (Dynamical equations)

$$
\begin{aligned}
\dot\mu_q + \dot\mu_p D_1 G(q,v) &= D_1 C(q,v,u) - \mu_p D_1 F(q,v,u),\\
\dot\mu_p D_2 G(q,v) &= D_2 C(q,v,u) - \mu_p D_2 F(q,v,u) - \mu_q,\\
0 &= D_3 C(q,v,u) - \mu_p D_3 F(q,v,u),\\
\dot q &= v,\\
D_1 G(q,v)\dot q + D_2 G(q,v)\dot v &= F(q,v,u).
\end{aligned}
$$

Discrete optimal control

Discrete Hamilton-Pontryagin action

$$
\begin{aligned}
\mathcal{J}_d = \sum_{k=0}^{N-1} \sum_{i=1}^{s} h b_i \Bigg[& C\left(Q_k^i, V_k^i, U_k^i\right) \\
& + \left\langle \mathrm{M}_{Q,k}^i, \frac{Q_k^i - q_k}{h} - \sum_{j=1}^{s} a_{ij} V_k^j \right\rangle + \left\langle \mu_{q,k+1}, \frac{q_{k+1} - q_k}{h} - \sum_{j=1}^{s} b_j V_k^j \right\rangle \\
& + \left\langle \mathrm{M}_{P,k}^i, \frac{D_2 L(Q_k^i, V_k^i) - D_2 L(q_k, v_k)}{h} - \sum_{j=1}^{s} \hat{a}_{ij} \left[D_1 L(Q_k^j, V_k^j) + f(Q_k^j, V_k^j, U_k^j) \right] \right\rangle \\
& + \left\langle \mu_{p,k+1}, \frac{D_2 L(q_{k+1}, v_{k+1}) - D_2 L(q_k, v_k)}{h} - \sum_{j=1}^{s} \hat{b}_j \left[D_1 L(Q_k^j, V_k^j) + f(Q_k^j, V_k^j, U_k^j) \right] \right\rangle \Bigg]
\end{aligned}
$$

Taking variations we find discrete necessary conditions for optimality.

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The setting

Nonholonomic Lagrangian systems

(L, Q, N), where $N \subset TQ$ constraint manifold. Locally described by null-set of $\Phi : TQ \to \mathbb{R}^m$, $m = \operatorname{codim}_{TQ} N$.

Dynamical equations

Obtained via Chetaev's principle, not variational.

$$\begin{aligned} \dot{q} &= v, \\ \dot{p} &= D_1 L(q, v) + \langle \lambda, D_2 \Phi(q, v) \rangle , \\ p &= D_2 L(q, v), \\ 0 &= \Phi(q, v). \end{aligned}$$

where λ Lagrange multipliers.

The setting (continued)

Nonholonomic controlled Lagrangian systems

(L, Q, N, f, U), where $N \subset TQ$ constraint manifold. Locally described by null-set of $\Phi : TQ \to \mathbb{R}^m$, $m = \operatorname{codim}_{TQ} N$.

Dynamical equations

Obtained via Chetaev's + D'Alembert's principle.

$$\begin{aligned}
\dot{q} &= v, \\
\dot{p} &= D_1 L(q, v) + \langle \lambda, D_2 \Phi(q, v) \rangle + f(q, v, u), \\
p &= D_2 L(q, v), \\
0 &= \Phi(q, v).
\end{aligned}$$

where λ Lagrange multipliers.

From constrained to free and forced

Lagrange multipliers

If the system is regular, then it is possible to differentiate $\Phi = 0$ w.r.t. t, substitute equations and solve for λ, leading to

$$\lambda(t) = l(q, v), \quad \text{or} \quad \lambda(t) = l(q, v, u)$$

It ensures every initial condition on N remains on N. This transforms the constrained system into a free forced system.

Controlled nonholonomic eqs. of motion still define section of $TT^*Q \times U$, of the form $(q, G(q, v), v, F(q, v, u), u)$ with $F(q, v, u) = D_1 L(q, v) + \langle l(q, v, u), D_2\Phi(q, v)\rangle + f(q, v, u)$.

Nonholonomic integrator

For Lobatto-type RK methods with (a_{ij}, b_j):

Nonholonomic integrator (finally on arxiv:1810.10926)

$$
\begin{aligned}
q_{k+1} &= q_k + h\sum_{i=1}^{s} b_i V_k^i, & D_2L(q_{k+1}, v_{k+1}) &= D_2L(q_k, v_k) + h\sum_{i=1}^{s} \hat{b}_i \left[D_1L(Q_k^j, V_k^j) + \left\langle \Lambda_k^j, D_2\Phi(Q_k^j, V_k^j) \right\rangle \right], \\
Q_k^i &= q_k + h\sum_{j=1}^{s} a_{ij} V_k^j, & D_2L(Q_k^i, V_k^i) &= D_2L(q_k, v_k) + h\sum_{j=1}^{s} \hat{a}_{ij} \left[D_1L(Q_k^j, V_k^j) + \left\langle \Lambda_k^j, D_2\Phi(Q_k^j, V_k^j) \right\rangle \right], \\
q_k^i &= Q_k^i, & D_2L(q_k^i, v_k^i) &= D_2L(q_k, v_k) + h\sum_{j=1}^{s} a_{ij} \left[D_1L(Q_k^j, V_k^j) + \left\langle \Lambda_k^j, D_2\Phi(Q_k^j, V_k^j) \right\rangle \right],
\end{aligned}
$$

$$0 = \Phi(q_k^i, v_k^i).$$

Hamiltonian flow:
$F_{L_d}^{\Lambda} : TQ|_N \times \Lambda \to TQ|_N \times \Lambda, (q_k, v_k, \lambda_k) \mapsto (q_{k+1}, v_{k+1}, \lambda_{k+1}),$
with $\lambda_k = \Lambda_k^1, \lambda_{k+1} = \Lambda_k^s$.

Lagrange multipliers in the discrete setting

Here comes trouble!

- If $\Lambda_k^i \mapsto l(Q_k^i, V_k^i, U_k^i)$, constraint not preserved in general!

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Here comes trouble!

- If $\Lambda_k^i \mapsto I(Q_k^i, V_k^i, U_k^i)$, constraint not preserved in general!
- Cannot easily solve for $\{\Lambda_k^i\}_{i=2}^s$, contrary to continuous case!

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- If solved, $\{\Lambda_k^i\}_{i=2}^s$ must be a function of $\lambda_k = \Lambda_k^1$ too!

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- Cannot easily solve for $\{\Lambda_k^i\}_{i=2}^s$, contrary to continuous case!
- If solved, $\{\Lambda_k^i\}_{i=2}^s$ must be a function of $\lambda_k = \Lambda_k^1$ too!

Solution

Imposing variations on $\{\Lambda_k^i\}_{i=2}^s$ appropriately.

Can also be done in the continuous case, but *a priori* might seem overkill.

Treating multipliers as independent variables

Continuous Hamilton-Pontryagin action

$$\mathcal{J}_{\mathcal{HP}} = \int_0^T \Big[C(q(t), v(t), u(t)) + \langle (\mu_q, \mu_p), (\dot{q} - v, (G(q,v))^{\cdot} - F(q,v,u,\lambda)) \rangle + \langle \mu_\lambda, \lambda - I(q,v,u) \rangle \Big] \, \mathrm{d}t$$

Treating multipliers as independent variables (continued)

Necessary conditions for optimality (Dynamical equations)

$$\begin{aligned}
\dot{\mu}_q + \dot{\mu}_p D_1 G(q,v) &= D_1 C(q,v,u) - \mu_p D_1 F(q,v,u,\lambda) - \mu_\lambda D_1 l(q,v,u), \\
\dot{\mu}_p D_2 G(q,v) &= D_2 C(q,v,u) - \mu_p D_2 F(q,v,u,\lambda) - \mu_\lambda D_2 l(q,v,u) - \mu_q, \\
0 &= D_3 C(q,v,u) - \mu_p D_3 F(q,v,u,\lambda) - \mu_\lambda D_3 l(q,v,u), \\
0 &= \mu_\lambda - \mu_p D_4 F(q,v,u,\lambda), \\
\dot{q} &= v, \\
D_1 G(q,v)\dot{q} + D_2 G(q,v)\dot{v} &= F(q,v,u,\lambda), \\
\lambda &= l(q,v,u).
\end{aligned}$$

Treating multipliers as independent variables (continued)

Necessary conditions for optimality (Dynamical equations)

$$\begin{aligned}
\dot{\mu}_q + \dot{\mu}_p D_1 G(q,v) &= D_1 C(q,v,u) - \mu_p D_1 F(q,v,u,\lambda) - \mu_\lambda D_1 l(q,v,u),\\
\dot{\mu}_p D_2 G(q,v) &= D_2 C(q,v,u) - \mu_p D_2 F(q,v,u,\lambda) - \mu_\lambda D_2 l(q,v,u) - \mu_q,\\
0 &= D_3 C(q,v,u) - \mu_p D_3 F(q,v,u,\lambda) - \mu_\lambda D_3 l(q,v,u),\\
0 &= \mu_\lambda - \mu_p D_4 F(q,v,u,\lambda),\\
\dot{q} &= v,\\
D_1 G(q,v)\dot{q} + D_2 G(q,v)\dot{v} &= F(q,v,u,\lambda),\\
0 &= \Phi(q,v).
\end{aligned}$$

Only need derivatives of $l(q,v,u)$. Equivalent to imposing $\delta\lambda = D_1 l(q,v,u)\delta q + D_2 l(q,v,u)\delta v + D_3 l(q,v,u)\delta u$.

Lagrange multipliers in the discrete setting

Discrete Lagrange multiplier variations

Casting nonholonomic integrator in the form

$$\mathcal{F}(x, y) = 0,$$

where $x = (q_k, v_k, \lambda_k, U_k^1, ..., U_k^s)$, $y = (q_{k+1}, v_{k+1}, \lambda_{k+1}, \Lambda_k^2, ...$
$..., \Lambda_k^{s-1}, ...)$, then

$$\delta y = -\left(D_2\mathcal{F}(x, y)\right)^{-1} D_1\mathcal{F}(x, y)\delta x$$

We get $\left\{\delta\Lambda_k^i\right\}_{i=2}^{s}$ in terms of $\delta q_k, \delta v_k, \delta\lambda_k, \left\{\delta U_k^i\right\}_{i=1}^{s}$.

Discrete nonholonomic optimal control

Discrete Hamilton-Pontryagin action

$$\mathcal{J}_d =$$

$$\sum_{k=0}^{N-1}\sum_{i=1}^{s} hb_i \Bigg[C\left(Q_k^i, V_k^i, U_k^i\right) + \left\langle \mathrm{M}_{Q,k}^i, \frac{Q_k^i - q_k}{h} - \sum_{j=1}^{s} a_{ij} V_k^j \right\rangle + \left\langle \mu_{q,k+1}, \frac{q_{k+1} - q_k}{h} - \sum_{j=1}^{s} b_j V_k^j \right\rangle$$

$$+ \left\langle \mathrm{M}_{P,k}^i, \frac{D_2 L(Q_k^i, V_k^i) - D_2 L(q_k, v_k)}{h} - \sum_{j=1}^{s} \hat{a}_{ij} \left[D_1 L(Q_k^j, V_k^j) + f(Q_k^j, V_k^j, U_k^j) + \Lambda_k^j D_2 \Phi(Q_k^j, V_k^j) \right] \right\rangle$$

$$+ \left\langle \mu_{p,k+1}, \frac{D_2 L(q_{k+1}, v_{k+1}) - D_2 L(q_k, v_k)}{h} - \sum_{j=1}^{s} \hat{b}_j \left[D_1 L(Q_k^j, V_k^j) + f(Q_k^j, V_k^j, U_k^j) + \Lambda_k^j D_2 \Phi(Q_k^j, V_k^j) \right] \right\rangle \Bigg]$$

Imposition of variations of Lagrange multipliers

$$\langle \delta \mathcal{J}_d, \delta c_d \rangle + \sum_{k=0}^{N-1}\sum_{i=2}^{s} hb_i \left\langle \mathrm{M}_{\Lambda,k}^i, \delta \Lambda_k^i - R_k^{q,i}\delta q_k - R_k^{v,i}\delta v_k - R_k^i \delta\lambda_k - \sum_{j=1}^{s} R_k^{U^j,i}\delta U_k^j \right\rangle$$

Discrete nonholonomic optimal control II

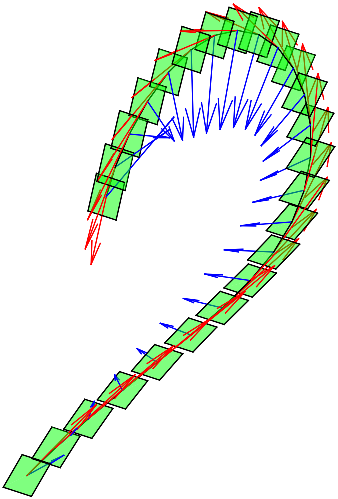
Closing the system

Resulting necessary conditions system must be supplemented with

$$\begin{aligned} D_2 L(q_k^i, v_k^i) &= D_2 L(q_k, v_k) \\ &\qquad + h \sum_{j=1}^{s} a_{ij} \left[D_1 L(Q_k^j, V_k^j) + f(Q_k^j, V_k^j, U_k^j) + \Lambda_k^j D_2 \Phi(Q_k^j, V_k^j) \right], \\ q_k^i &= Q_k^i, \\ 0 &= \Phi(q_k^i, v_k^i). \end{aligned}$$

Example: Nonholonomic controlled particle

$$
\begin{aligned}
Q &= \mathbb{R}^3 \\
U &= \mathbb{R}^2 \\
L(q, v) &= \frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right) \\
\Phi(q, v) &= v_z - y v_x \\
f(q, v, u) &= u^1 \mathrm{d}x + u^2 \mathrm{d}y + y u^1 \mathrm{d}z \\
C(q, v, u) &= \frac{1}{2}\left(u_1^2 + u_2^2\right).
\end{aligned}
$$



THANKS FOR YOUR ATTENTION!