Lie–Poisson simulation of the dynamics of point vortices on a rotating sphere coupled with a background field

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One of the major unsolved problems of the Euler equations on a sphere is the long-time behaviour of an inviscid fluid when a certain initial vorticity is given [4],[6]. The biggest challenge in understanding and simulating this system is due to the presence of an infinite number of first integrals (Casimir functions), which however don't provide complete integrability [7]. Moreover, the Hamiltonian nature of the equations suggests that much geometry is involved in them. They are in fact a Lie–Poisson system on $sdiff^*(\mathbb{S}^2)$ (the dual of the Lie algebra of divergence free vector fields) [1]. Special solutions to the Euler equations come from the fact that on $sdiff^*(\mathbb{S}^2)$ there exist non trivial finite dimensional coadjoint orbits, called *point vortices* [5]. However, this orbits have physical interest only when the sphere is non-rotating. When this is not true the Euler equations become a coupled system of equations of a singular field of point vortices and a smooth continuous background vorticity [3].

Starting from [2],[8], we present an approximation of the coupled model based on the quantization of Kähler manifolds, which keeps the Hamiltonian Lie-Poisson structure of the equations. Moreover, with the techniques of the geometric integration, we provide a Lie-Poisson numerical scheme to solve the quantized model, preserving up to roundoff precision the discrete Casimirs and, up to the order of the method, the Hamiltonian. The conservation of the quantized first integrals provides a deep insight in the nature of the Euler equations and a better qualitative simulation of them. Furthermore, our numerical scheme provides a useful tool in studying the still unknown persistence of relative equilibria of point vortices passing from a non-rotating to a rotating sphere.

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