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Group 11

Oscillations in Gravity-Driven Fluid Exchange

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Abstract

If you take a bottle with water and turn it upside down, the water will spill out. But as it spills, air comes in. A little bit of experimentation shows that if the opening in the bottle is narrow enough, the air comes in in bursts, bubble by bubble. This sometimes causes a periodic sound, the characteristic "glug-glug" of the water coming out. The aim of the project is to experiment with the system and then critically consider and possibly improve, by incorporating the results of experimentation, existing models for the process.

11.1 Introduction

The glug-glug effect, in fluid dynamics, is the characteristic sound produced by the alternation of a fluid coming out and air going in through an opening of a container, when it is turned upside down. It is periodic, very distinctive and accompanied by bubbles of air. It is also the subject of our work. We consider an axisymmetrical bottle with one opening filled with water being emptied while held static at 180° (opening down). We must take into account the dimensions of the bottle and of the opening, the initial quantity of water, initial pressures of water and air. We have analyzed existing models, as well as suggesting a modelling framework of our own. We obtain experimental results and use them to guide our intuition and as a basis for evaluating theoretical models.

The structure of this document is as follows: the remainder of this section is devoted to presenting the basics of fluid mechanics; in Section 11.2 we present the experimental setup; in the following two sections we analyse the models of Clanet [1] and Kohira [3], and finally in Section 11.5 we draw our conclusions and make suggestions for further work.

Continuity equation

Regardless of the flow assumptions, a statement of the conservation of mass is generally necessary. This is achieved through the mass continuity equation [1], given in its most general form as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{11.1}$$

or, using the material derivative:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0. \tag{11.2}$$

where $\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$.

Navier-Stokes equations

The NavierStokes equations, named after Claude-Louis Navier and George Gabriel Stokes, describes the motion of viscous fluid substances.

These equations are a special case of conservation of momentum. In a very basic sense, they are derived from Newton's second law, $F = ma$ which for non-relativistic systems is accurate.

[1] See `https://en.wikipedia.org/wiki/Continuity_equation`

The first assumption we need to make is that the fluid is continuous, which is wrong since everything is made of discrete particles, but for anything large enough, it's a good assumption. By assuming this, we can transform Newton's second law into the Cauchy momentum equation [2]:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla\cdot P + \mathbf{F_b} \tag{11.3}$$

This equation is equivalent to Newton's second law. The left-hand side is mass times acceleration (taking into account the convective acceleration [3] and the right-hand side is the sum of the forces exerted on a point (the divergence of the stress tensor $\nabla \cdot P$, i.e. how the stress field is changing, and F_b is the sum of body forces like weight due to gravity or friction). At this point, all we have done was to make one assumption and a little maths. Note: we are also assuming an inertial frame of reference at the moment to simplify the equation.

The key to the Navier-Stokes equations is that they assume a constitutive equation [4]. A constitutive equation relates two quantities (in this case force and velocity) for a given material.

The general expression for the Navier-Stokes equation is:

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{F_b} - \nabla P + \mu\left(\frac{1}{3}\nabla\left(\nabla\cdot\mathbf{u}\right) + \nabla^2\mathbf{u}\right) \tag{11.4}$$

where $\mathbf{u}$ is the velocity of the fluid, P the total pressure acting on the system, μ the viscosity coefficient and $\mathbf{F_b}$ the forces acting over the fluid.

Air and Water

Air: As said above, we assumed air to act like an ideal gas, so the ideal gas law[5] could be used, $PV = nRT$, where P is pressure, V is volume of air, n the number of moles, R the ideal gas constant and T the temperature. This is important for the study of the bubble formed upon the entrance of air after surface tension breaks at the opening of the bottle. We assumed pressure outside the bottle to be the atmospheric pressure.

Water: Water was considered as an inviscid, incompressible fluid, which implies $\mu = 1$ and ρ to be constant in the Navier-Stokes equations, which then leads to $\nabla \cdot \mathbf{u} = 0$. The forces acting on the water are gravity, pressure from the mass of water within the bottle and from the air (both inside and outside the bottle).

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[2] See `https://en.wikipedia.org/wiki/Cauchy_momentum_equation`

[3] See `https://en.wikipedia.org/wiki/Cauchy_momentum_equation\#Convective_acceleration`

[4] See `https://en.wikipedia.org/wiki/Constitutive_equation`

[5] see `https://en.wikipedia.org/wiki/Ideal_gas_law`

11.2 Experiments

In this section, the experimental setup will be described.

Problem description and its physical dependence

Recall that our problem consists of the emptying of a cylindrical tank filled of liquid into the atmosphere. As an initial condition of the problem, we assume the tank is half-full of an incompressible fluid and the remaining space of the tank is full of gas, a compressible fluid.

Emptying is carried through a small hole at the bottom of the tank. The hole is small enough so that the flow is not dominated by significant counterflow of the two liquids. For a diagram, please see Figure 11.3.

In the general problem, we would like to compute two quantities. The first one is the total emptying time,T_e. The other one is the period of a cycle, T. The cycle is compounded of the emptying of some liquid and the generation of one bubble of gas. Our problem involves a number of parameters.

These parameters are of two kinds. On the one hand, we have geometric parameters, $GP = (L,\ D,\ d,\ z_0,\ z_i)$. On the other hand, we have physical characteristics of the system. $PC = (g,\ P_0,\ \rho_a,\ \rho_w,\ \nu_a,\ \nu_w,\ \sigma,\ \beta)$. Specifically, we have:

- L, total height of the tank.
- D, diameter of the tank.
- d, diameter of the opening of the tank
- z_0, initial height of the liquid fluid.
- z_i, instantaneous height of the liquid.
- g, gravity.
- P_0, initial pressure of the gas.
- ρ_a, gas density.
- ρ_w, liquid density.
- ν_a, gas kinematic viscosity.
- ν_w, liquid kinematic viscosity.
- σ, surface tension between both fluids.

- β, gas compressibility constant.

Thus

$$\begin{aligned} T_e &= \mathcal{F}(GP, PC) \\ T &= \mathcal{G}(GP, PC) \end{aligned} \tag{11.5}$$

Experiment restrictions

During the experiments, we have made a number of restrictions of the physics of the problem.

Our choice for the liquid is water. For gas, we have chosen air and the initial pressure for air is the atmospheric pressure at atmospheric temperature.

We are interested in an oscillating solution of the problem. If the initial height of water is close to the total height of the tank, the mass of the air is not sufficient to create an oscillating solution. In that case the problem only has a stationary solution, an equilibrium between the force exerted by the weight of water column and the force that comes from the integral of surface tension. A similar lack of a periodic solution has been observed when a quite small diameter of the hole of the tank is made.

In the opposite case, if we make a rather wide hole at the bottom of the tank, bubble generation will occur at the same time as the water discharge. So this case does not have a oscillating solution. The range of diameters where we have a oscillation solution is between $d \in [6, 16] mm$.

Our experiment was carried out with a bottle hall full of water with a small hole at the bottom of the bottle. The bottle was fixed with an improvised harness, see figure 11.1.

Results and conclusions

With the bottle half-full, and with the diameter of the opening chosen judiciously, we obtain the required oscillatory solution. Our ability to video experiments and to slow the video down, also allows us to make some novel observations. We will discuss them in 11.5. In brief, the mechanistic picture used by [1] and [3] though generally sound, is not necessarily absolutely correct.

Next, we want to present and criticise the approaches available in the literature.

Figure 11.1: Photography of our experiment

Figure 11.2: Experiment in progress

11.3 The Clanet-Searby Model

Introduction

The common experience of the emptying of a vertical bottle initially full of liquid, surrounded by air, and submitted to an acceleration due to gravity, reveals that the liquid flows out of the bottle through an alternating succession of jets of liquid and admissions of air bubbles. This oscillatory path back to the equilibrium is referred to by the onomatopoeic *glug-glug* and is characterized by the period of the oscillations T. This oscillatory behaviour starts at the opening and continues until the bottle is empty, that is all along the emptying time T_e. The life of the bottle can thus be characterized by the ratio T_e/T.

To understand the physical laws governing the existence of this system, we first reduce the problem to the emptying of a vertical cylinder of diameter D_0 and length L, closed at the top and open at the bottom through a circular thin-walled hole of diameter d, on the axis of the cylinder (figure 11.3). The cylinder being initially filled with a liquid. At $t = 0$ we open the hole d and look for the laws governing both T_e and T as a function of the interface location z_i. The study is performed in the low-viscosity limit.

In this section, we follow the approach of C. Clanet and G. Searby [1]. In particular, we show that the periodicity of motion is only obtained if a drastic and unjustified simplification is made.

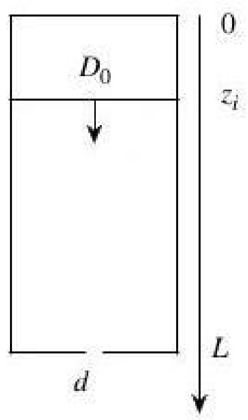

Figure 11.3: Experimental set-up and orientation of axes

On the long time scale T_e

To model the dynamics of the liquid interface on the long time scale T_e, we assume that the long and short time scales are decoupled, $T_e/T \gg 1$, so that on the long time scale the emptying phenomenon appears as continuous. Using the constraint of constant volume and some known laws about the velocity of bubbles it can be shown that

$$T_e \simeq \frac{2L}{\sqrt{gD_0}} \left(\frac{D_0}{d} \right)^{5/2} .$$

This time is independent of the liquid properties (density, viscosity and surface tension with the surrounding air) and for a given diameter of hole d, it increases with the volume of the bottle $\propto D_0^2 L$.

On the short time scale T

As for the short time scale, experimentally it can be shown that the physical origin of the oscillations of the *glug-glug* lies in the compressibility of the surrounding gas. Using a spring-mass analogy, it can be demonstrated that, letting $\bar{z}_i$ the mean position of the interface, the period T is equal to

$$T = 2\pi \frac{L}{\sqrt{\gamma P_0/\rho}} \sqrt{\frac{\bar{z}_i}{L} \left(1 - \frac{\bar{z}_i}{L} \right)}, \tag{11.6}$$

where γ is the isoentropic expansion factor, P_0 the atmospheric pressure and ρ the density of the liquid.

The continuous equations

We now derive in the low-viscosity limit the equations that govern the oscillations. In particular, the latter are decomposed into two phases: the outflow of the liquid and the admission of a new bubble without liquid flow. Since the emptying time T_e is large compared to the period T, we assume that, during the whole period, the mean position $\bar{z}_i$ of the interface is constant. Thus the actual interface location z_i is decomposed into a fixed part $\bar{z}_i$ and a time-dependent part $\tilde{z}_i$ i.e. $z_i(t) = \bar{z}_i + \tilde{z}_i(t)$. To describe the motion of the liquid in the tube during the whole cycle we use the Euler's equation

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \mathbf{U} \cdot \mathbf{U} = -\frac{1}{\rho} \nabla p + \mathbf{g}. \tag{11.7}$$

Projecting (11.7) onto a streamline and integrating between two points A and B, respectively located on the upper interface and the exit of the tube, we obtain:

$$\int_A^B \frac{\partial \mathbf{U}}{\partial t} \cdot dl + \frac{1}{2}(U_B^2 - U_A^2) = -\frac{1}{\rho}(P_B - P_A) + g(L - z_i) \tag{11.8}$$

Assuming an isoentropic transformation, the right-hand side of (11.8) can be linearized to yield

$$-\frac{1}{\rho}(P_B - P_A) + g(L - z_i) = -\frac{\gamma P_0}{\rho \bar{z}_i}\tilde{z}_i\left[1 - \lambda + \lambda\left(\frac{\bar{z}_i}{L}\right)\left(1 + \frac{1}{\gamma}\right)\right], \tag{11.9}$$

where $\lambda = \rho g L / P_0$. Let $F(\lambda, \bar{z}_i)$ the expression in square brackets of (11.9). During the outflow it can be deduced that

$$\frac{1}{2}(U_B^2 - U_A^2) = \frac{1}{2}\left[\left(\frac{D_0}{d}\right)^4 - 1\right]\dot{\tilde{z}}_i^2,$$

while the acceleration term in (11.8) can be approximated

$$\int_A^B \frac{\partial \mathbf{U}}{\partial t} \cdot dl \approx \left[L - \bar{z}_i + \sqrt{\alpha}\frac{D_0^2}{d}\left(1 - 2\frac{d}{D_0}\right)\right]\ddot{\tilde{z}}_i = (L - \bar{z}_i + \mathscr{L})\ddot{\tilde{z}}_i,$$

where α is a constant. The Euler equation (11.7) during the outflow can thus be written

$$(L - \bar{z}_i + \mathscr{L})\ddot{\tilde{z}}_i + \frac{1}{2}\left[\left(\frac{D_0}{d}\right)^4 - 1\right]\dot{\tilde{z}}_i^2 + \frac{\gamma P_0}{\rho \bar{z}_i}\tilde{z}_i F(\lambda, \bar{z}_i) = 0 \tag{11.10}$$

During the entry of the bubble into the tube (i.e. the inflow), the nonlinear term disappears and the dynamics of the interface is described by the equation

$$(L - \bar{z}_i)\ddot{\tilde{z}}_i - \frac{1}{2}\dot{\tilde{z}}_i^2 + \frac{\gamma P_0}{\rho \bar{z}_i}\tilde{z}_i F(\lambda, \bar{z}_i) = 0. \tag{11.11}$$

Since we look for small oscillations around the equilibrium, in both equations (11.10)-(11.11) we retain only the linear terms. The dynamics of the interface during the whole cycle is thus described by the system

$$\ddot{\tilde{z}}_i + \frac{\gamma P_0}{\rho L^2}\frac{F(\lambda, \bar{z}_i)}{\bar{z}_i / L}\frac{1}{1 - \bar{z}_i/L + \mathscr{L}/L}\tilde{z}_i = 0, \tag{11.12}$$

$$\ddot{\tilde{z}}_i + \frac{\gamma P_0}{\rho L^2}\frac{F(\lambda, \bar{z}_i)}{\bar{z}_i / L}\frac{1}{1 - \bar{z}_i/L}\tilde{z}_i = 0, \tag{11.13}$$

where the first equation describes the outflow $\dot{\tilde{z}}_i > 0$ and the second the inflow $\dot{\tilde{z}}_i < 0$ (cf. the orientation of the axes in figure 11.3). This linearization of the problem enables to find an analytical expression of the period T of the oscillations, that is compatible with the period given by the spring-mass analogy (11.6).

Criticism

In the previous analysis there are some points which aren't completely right.

The authors obtain (11.12) by neglecting the nonlinear term of the equation (11.10). But the expression in brackets

$$\frac{1}{2}\left[\left(\frac{D_0}{d}\right)^4 - 1\right] \tag{11.14}$$

is very large because $D_0 \gg d$. For instance, if we take $D_0 = 78.9$ mm and $d = 10$ mm we obtain in (11.14) more than 1937.

As for the entire cycle, it is governed by the system of equations (11.12)-(11.13). But each equation characterizes a simple harmonic oscillator too, which has by its nature periodic solutions.

Making simulations in Matlab with the complete system (11.10)-(11.11), we have never found any periodic solutions. In figure 11.4 we report the phase plane, with the physical parameters in caption. We recognize two different behaviours. In the lower half-plane the orbits are semi-elliptic, due to equation (11.11), while in the upper one they are curves that start and end in points which aren't symmetric. With this behaviour the orbit collapses towards the origin.

So, on the one hand if we keep the nonlinear term, we don't find periodic solutions and the *glug-glug* regime doesn't begin; on the other hand if we neglect that term, we find periodic solutions but the deletion of that term is not allowed.

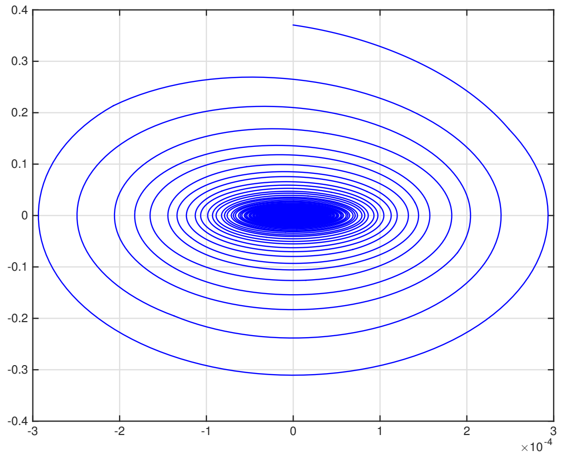

Figure 11.4: Phase plane; $D_0 = 86$ mm, $d = 14$ mm, $L = 300$ mm

11.4 The Kohira Model

In this section, we re-derive the model of the Japanese physicists Kohira *et. al* [3] without linearization in order to see if we obtain different results.

This model considers (plastic or glass) axisymmetric bottle nearly full of water and with some volume V of air inside. The bottle will be always in the position showed in Figure 11.4.

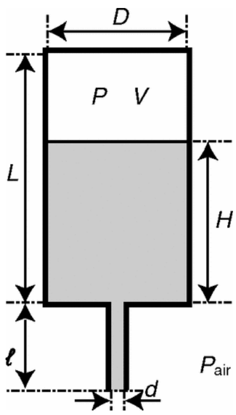

Figure 11.5: System scheme.

Where:

- D is the diameter of the circular base of the bottle.
- d is the diameter of opening of the bottle.
- l is the length of the bottle's neck.
- L is the length of bottle's body.
- H is the distance between the base and the upper part of the bottle's neck.
- P_{air}, which we will refer to as P_a, is the pressure of the air that is outside the bottle.
- P is the pressure of the amount of air that is inside the bottle.

First we will analyse the downflow of water finding the equations that describe the process. After this we do the same with the upflow of air bubbles.

The assumptions made are essentially the following:

- The fluid we are working with (water) is assumed to be incompressible, which implies that

$$\boxed{\rho = constant}$$

this condition also implies (as a consequence of the divergence theorem) that

$$\boxed{\nabla \cdot \mathbf{u} = 0}$$

- We assume an isothermal process.
- We assume air to be an ideal gas.

After this assumptions are made, the Navier-Stokes equation (11.4) for the velocity of water (resp. air) during the downflow (resp. overflow) acquires the following form:

$$\rho \frac{\partial u}{\partial t} = -\gamma_f u - \nabla \cdot P + sgn(u)\rho g \tag{11.15}$$

where γ_f is a coefficient that takes into account the viscosity of the fluid so this term corresponds to a frictional force. We will use w and v_w instead of f and u for water, and a, v_a when we study air. We also have taken the upper direction as positive.

Phase 1: downflow of water.

As Kohira *et al.* did in [3], we consider the differential equation for the average velocity of water coming out, $v_w = v_w(t)$, which is derived from Navier-Stokes equations:

$$\rho \frac{dv_w}{dt} = -\gamma_w v_w + \frac{P_a - \rho g H - P}{l} - \rho g \tag{11.16}$$

where the $\rho g H/l$ term is due to the water that is above the bottle's neck, g is the gravity and ρ is the density of water. The assumption that air is an ideal gas give us the relation between P and V:

$$PV = N k_B T \tag{11.17}$$

where N is the number of molecules of air, k_B ($\approx 1.38065 \times 10^{-23} \frac{m^2 Kg}{s^2 K}$) the Boltzmann constant and T the temperature of the air.

We now want to write the parameters v_w, $\frac{dv_w}{dt}$ and H as functions of the pressure P:

In this case, because there is no inflow or outflow of air during the downflow of water, N remains constant. Moreover, if we also assume T to be constant, then we have that $Nk_BT = PV$ is constant. By differentiating equation (11.17) with respect to time we have

$$0 = \frac{d}{dt}(PV) = P\frac{dV}{dt} + V\frac{dP}{dt} \implies -\frac{dV}{dt} = \frac{V}{P}\frac{dP}{dt}$$

and using equation (11.17) we obtain

$$-\frac{dV}{dt} = \frac{Nk_BT}{P^2}\frac{dP}{dt} \tag{11.18}$$

Now continuity equation (conservation of mass) gives us the relation $A_1v_1 = A_2v_2$ between two points, 1 and 2, that are in the same stream line of the fluid, where v_i are velocities and A_i are sections (in our case horizontal sections of the bottle). So we have

$$\pi\left(\frac{d}{2}\right)^2 v_w = \frac{Nk_BT}{P^2}\frac{dP}{dt}$$

which gives us a relation between v_w and the pressure P and $\dot{P}$:

$$v_w = \left(\frac{2}{d}\right)^2 \frac{Nk_BT}{\pi}\frac{1}{P^2}\frac{dP}{dt} = \beta\frac{1}{P^2}\dot{P} \tag{11.19}$$

where we have defined $\beta := \left(\frac{2}{d}\right)^2 \frac{Nk_BT}{\pi}$ for simplicity.

By differentiating with respect to time equation (11.19) we obtain $\frac{dv_w}{dt}$ as a function of P, $\dot{P}$ and $\ddot{P}$:

$$\begin{aligned}\frac{dv_w}{dt} &= \beta\left[\frac{d}{dt}\left(\frac{1}{P^2}\right)\frac{dP}{dt} + \frac{1}{P^2}\frac{d^2P}{dt^2}\right]\\ &= \beta\left[\frac{-2}{P^3}\left(\frac{dP}{dt}\right)^2 + \frac{1}{P^2}\frac{d^2P}{dt^2}\right]\end{aligned}$$

so we obtain

$$\frac{dv_w}{dt} = \frac{\beta}{P^2}\left(\ddot{P} - 2\frac{\dot{P}^2}{P}\right) \tag{11.20}$$

Finally we search for an expression of H as a function of P and its derivatives.

As $V = (L - H)A$, where A is the area of the base of the bottle, we have

$$H = L - \frac{V}{\left(\frac{D}{2}\right)^2 \pi}$$

and using equation (11.17) we obtain

$$H = L - \left(\frac{2}{D}\right)^2 \frac{Nk_BT}{\pi}\frac{1}{P} \tag{11.21}$$

Now using equations (11.19), (11.20) and (11.21), replacing them in equation (11.16) and reordering terms, we obtain the following differential equation for the pressure P:

$$\begin{aligned}&\ddot{P}-2\frac{\dot{P}^2}{P}+\frac{\gamma_w}{\rho}\dot{P}+\frac{1}{l\rho\beta}P^3\\&-\frac{1}{\beta}\left(\frac{P_a}{l\rho}-g\left(1+\frac{L}{l}\right)\right)P^2-\frac{g}{l}\left(\frac{d}{D}\right)^2P=0.\end{aligned}\tag{11.22}$$

To achieve a non-dimensionalized and normalized description, firstly we write equation (11.22) in terms of $x=\frac{P}{P_a}$ obtaining

$$\begin{aligned}&\ddot{x}-2\frac{\dot{x}^2}{x}+\frac{\gamma_w}{\rho}\dot{x}+\frac{P_a^2}{l\rho\beta}x^3\\&-\frac{P_a}{\beta}\left(\frac{P_a}{l\rho}-g\left(1+\frac{L}{l}\right)\right)x^2-\frac{g}{l}\left(\frac{d}{D}\right)^2x=0.\end{aligned}\tag{11.23}$$

Now we rescale the variable t by $\sqrt{\frac{g}{l}}t\to t$, $\sqrt{\frac{g}{l}}\frac{d}{dt}\to\frac{d}{dt}$, $\frac{g}{l}\frac{d^2}{dt^2}\to\frac{d^2}{dt^2}$ to obtain the non-dimensionalized equation we wanted:

$$\boxed{\ddot{x}-2\frac{\dot{x}^2}{x}+\frac{\gamma_w}{\rho}\sqrt{\frac{l}{g}}\dot{x}+\frac{P_a^2}{g\rho\beta}x^3-\frac{lP_a}{g\beta}\left(\frac{P_a}{l\rho}-g\left(1+\frac{L}{l}\right)\right)x^2-\left(\frac{d}{D}\right)^2x=0}\tag{11.24}$$

Defining $A_1:=\left(\frac{d}{D}\right)^2$, $A_2:=\frac{lP_a}{g\beta}\left(\frac{P_a}{l\rho}-g\left(1+\frac{L}{l}\right)\right)$, $A_3:=\frac{P_a^2}{g\rho\beta}$ and $A_4:=\frac{\gamma_w}{\rho}\sqrt{\frac{l}{g}}$ (all positive constants if we use realistic values for all the parameters), we can write equation (11.24) as

$$\ddot{x}-2\frac{\dot{x}^2}{x}+A_4\dot{x}+A_3x^3-A_2x^2-A_1x=0\tag{11.25}$$

Phase 2: upflow of air.

As we had for the downflow of water, now the equation for the average velocity of air coming up is:

$$\rho\frac{dv_a}{dt}=-\gamma_a v_a+\frac{P_a-\rho gH-P}{l}+\rho g\tag{11.26}$$

The term ρg corresponds to the floating force as air exists in the water.

In this case we have that V is constant because of the incompressibility of water, but now N is changing as air molecules come into the bottle. By differentiating equation (11.17) with respect to time, we have

$$P\cancel{\frac{dV}{dt}} + V\frac{dP}{dt} = k_B T\frac{dN}{dt} \implies \frac{dP}{dt} = \frac{k_B T}{V}\frac{dN}{dt} \tag{11.27}$$

From the conservation law for air,

$$\frac{dN}{dt} = \rho_a \pi\sigma \left(\frac{d}{2}\right)^2 v_a \tag{11.28}$$

where ρ_a is the molecule number density of the air and σ is the volume fraction of the bubble to the bottle.

As we did in the previous subsection, we search for expressions of H, v_a and $\frac{dv_a}{dt}$ as functions of P and its derivatives. For H equation (11.21) is still valid, but now what is constant is the volume V, not N, so that using the ideal gas relation (11.17) we obtain a new expression for H:

$$H = L - \left(\frac{2}{D}\right)^2 \frac{V}{\pi} \tag{11.29}$$

Now using equations 11.27 and 11.28 we have that

$$\frac{dP}{dt} = \frac{k_B T}{V}\rho_a \pi\sigma \left(\frac{d}{2}\right)^2 v_a$$

obtaining

$$v_a = \frac{V}{k_B T \rho_a \pi\sigma}\left(\frac{2}{d}\right)^2 \frac{dP}{dt} = \eta\frac{dP}{dt} \tag{11.30}$$

where $\eta = \frac{V}{k_B T \rho_a \pi\sigma}\left(\frac{2}{d}\right)^2$. Finally differentiating equation 11.26 with respect to time, we obtain the equation for $\frac{dv_a}{dt}$:

$$\frac{dv_a}{dt} = \eta\frac{d^2 P}{dt^2} \tag{11.31}$$

Plugging equations 11.29, 11.30 and 11.31 into equation 11.26 and reordering terms, we obtain the following differential equation for the pressure:

$$\begin{aligned} &\frac{d^2P}{dt^2} + \frac{\gamma_a}{\rho}\frac{dP}{dt} + \frac{1}{\rho\eta l}P \\ &- \left[\frac{P_a}{l\rho\eta} + \frac{g}{l\eta}\left(L - \left(\frac{2}{D}\right)^2 \frac{V}{\pi}\right) + \frac{g}{\eta}\right] = 0 \end{aligned} \tag{11.32}$$

To achieve a non-dimensionalized and normalized description, firstly we write equation (11.32) in terms of $x = \frac{P}{P_a}$ obtaining

$$\begin{aligned} &\frac{d^2x}{dt^2} + \frac{\gamma_a}{\rho}\frac{dx}{dt} + \frac{1}{\rho\eta l}x \\ &- \left[\frac{1}{l\rho\eta} + \frac{g}{l\eta P_a}\left(L - \left(\frac{2}{D}\right)^2\frac{V}{\pi}\right) + \frac{g}{P_a\eta}\right] = 0 \end{aligned} \tag{11.33}$$

Now we rescale the variable t by $\sqrt{\frac{g}{l}}t \to t$, $\sqrt{\frac{g}{l}}\frac{d}{dt} \to \frac{d}{dt}$, $\frac{g}{l}\frac{d^2}{dt^2} \to \frac{d^2}{dt^2}$ to obtain the non-dimensionalized equation we wanted:

$$\boxed{\frac{d^2x}{dt^2} + \frac{\gamma_a}{\rho}\sqrt{\frac{l}{g}}\frac{dx}{dt} + \frac{1}{\rho\eta g}x - \frac{l}{g}\left[\frac{1}{l\rho\eta} + \frac{g}{l\eta P_a}\left(L - \left(\frac{2}{D}\right)^2\frac{V}{\pi}\right) + \frac{g}{P_a\eta}\right] = 0} \tag{11.34}$$

Defining $B_3 := \frac{\gamma_a}{\rho}$, $B_2 := \frac{1}{\rho\eta g}$ and $B_1 := \frac{l}{g}\left[\frac{1}{l\rho\eta} + \frac{g}{l\eta P_a}\left(L - \left(\frac{2}{D}\right)^2\frac{V}{\pi}\right) + \frac{g}{P_a\eta}\right]$ (all positive constants if we use realistic values for all the parameters), we can write equation (11.34) as

$$\frac{d^2x}{dt^2} + B_3\frac{dx}{dt} + B_2x - B_1 = 0 \tag{11.35}$$

which is clearly a linear differential equation with constant coefficients.

PHASE 1 + PHASE 2 mathematical description

Suppose we have our bottle prepared to carry out the experiment but with the hole covered. At the time $t_0 = 0$ state variables of our system take the initial values $P_0 = P(t_0)$, $V_0 = V(t_0)$ and $N_0 = N(t_0)$. At time t_0 we open the hole and a drop of water begins to form. For a time t_1 system will be in the first phase, so that the equation that governs the system is (11.24), where t_1 is defined as the minimum $t > 0$ such that $\dot{x}_0(t) = 0$ [6]. For $t \in [t_0, t_1]$ we have $N(t) = N(t_0)$ because during the outflowing of water N remains constant, while P and V are variable. For $t \in [t_0, t_1]$ we have that $P(t) = P_a x_0(t)$.

At the time t_1 the equation that describes our system, which is in the second phase, is equation (11.34), with initial conditions $N_1 = N(t_1) = N(t_0) = N_0$, $P_1 = P(t_1) = x_0(t_1)P_a$ and $V_1 = V(t_1)$ is determined by the relation $P_0V_0 = P_1V_1$. Until a certain minimum time $t_2 > 0$ such that $\dot{x}_1(t_2) = 0$

[6] Here $x_0(t)$ is the solution of equation (11.24) with the initial conditions $(P_0, \dot{P}_0 = 0, V_0, N_0)$ for $t \in [0, t_1]$, so $x_0(0) = P_0P_a$ and $\dot{x}_0(0) = 0$

[7], i.e. for $t \in [t_1, t_1 + t_2]$, we have $V(t) = V(t_1)$ because during the upflow of air, volume is considered to be constant, and N, P are now variable. In fact, now $P(t) = P_a x_1(t - t_1)$ for $t \in [t_1, t_1 + t_2]$.

At the time $t_1 + t_2$ the formation of a new drop of water starts, so we are again in the first phase. Now for a certain interval of time $[t_1 + t_2, t_1 + t_2 + t_3]$ the equation (11.24) and initial conditions $P_2 = P(t_1 + t_2) = x_1(t_2)P_a$, $V_2 = V(t_1 + t_2) = V(t_1) = V_1$, $N_2 = N(t_1 + t_2)$[8] will give us a solution $x_2(t)$, for $t \in [0, t_3]$ and t_3 the minimum $t > 0$ satisfying $\dot{x}_2(t_3) = 0$, which will determinate $P(t)/P_a$ for $t \in [t_1 + t_2, t_1 + t_2 + t_3]$.

We can repeat this process until the bottle has emptied, in a particular time t_n, to define a function $X(t) = \frac{P(t)}{P_a}$ that describes the evolution of the pressure during all time:

$$X(t) = \begin{cases} x_0(t), & t \in [t_0, t_1] \\ x_i\left(t - \sum_{j=1}^{i} t_j\right), & t \in \left[\sum_{j=1}^{i} t_j, \sum_{j=1}^{i+1} t_j\right] \quad \text{for } 1 \leq i \leq n-1. \end{cases}$$

On the one hand, $x_i(t)$ for $i \in 2\mathbb{Z}$, $i \leq n$, is the solution of equation (11.24) taking $x(0) = \frac{P_i}{P_a}$ and $\dot{x}(0) = 0$ as initial conditions and setting $\beta := \beta_i$ with $\beta_i = \left(\frac{2}{d}\right)^2 \frac{N_i k_B T}{\pi}$ where $N_i = N\left(\sum_{j=0}^{i} t_j\right)$ and $P_i = P\left(\sum_{j=0}^{i} t_i\right)$ for all $i \geq 0$ even. So $x_i(t)$, $t \in [0, t_i]$, solves the following Cauchy problem

$$\ddot{x}_i - 2\frac{\dot{x}_i^2}{x_i} + \frac{\gamma_w}{\rho}\sqrt{\frac{l}{g}}\dot{x}_i + \frac{P_a^2}{g\rho\beta_i}x_i^3 - \frac{lP_a}{g\beta_i}\left(\frac{P_a}{l\rho} - g\left(1 + \frac{L}{l}\right)\right)x_i^2 - \left(\frac{d}{D}\right)^2 x_i = 0,$$

$$x_i(0) = \frac{P_i}{P_a}, \quad \dot{x}_i(0) = 0.$$

On the other hand, $x_i(t)$ for $i \notin 2\mathbb{Z}$, $i \leq n$, is the solution of equation (11.34) taking again $x(0) = \frac{P(t_i)}{P_a}$, $\dot{x}(0) = 0$ as initial conditions and now replacing V by V_i, so that this time we have to redefine $\eta := \eta_i$ with $\eta_i = \frac{V_i}{k_B T \rho_a \pi \sigma}\left(\frac{2}{d}\right)^2$, where $V_i = V\left(\sum_{j=0}^{i} t_j\right)$ and P_i is defined as we did for $i \in 2\mathbb{Z}$. Now $x_i(t)$,

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[7] Here $x_1(t)$ is the solution of equation (11.34) with the initial conditions $(P_1, \dot{P}_1 = 0, V_1, N_1)$ for $t \in [0, t_2]$, so $x_1(0) = P_1 P_a$, $\dot{x}_1(0) = 0$ and $x_1(0) = x_0(t_1)$.

[8] Now we can determine N_2 using the relation $P_2 V_2 = P_2 V_1 = N_2 k_B T$

$t \in [0, t_i]$, solves the following Cauchy problem

$$\ddot{x}_i + \frac{\gamma_a}{\rho}\sqrt{\frac{l}{g}}\dot{x}_i + \frac{1}{\rho\eta_i g}x_i - \frac{l}{g}\left[\frac{1}{l\rho\eta_i} + \frac{g}{l\eta P_a}\left(L - \left(\frac{2}{D}\right)^2\frac{V_i}{\pi}\right) + \frac{g}{P_a\eta_i}\right] = 0,$$

$$x_i(0) = \frac{P_i}{P_a}, \quad \dot{x}_i(0) = 0.$$

To build algorithmically our solution $X(t)$ we need at each step the values for P_i, N_i and V_i. We can determine them as follows:

- For $i = 0$, N_0, V_0 and P_0 are determined empirically.
- For $i \in 2\mathbb{Z}$, $i \geq 2$, we have that $P_i = x_{i-1}(t_i)$, $V_i = V_{i-1}$, and $N_i = \frac{P_i V_i}{k_B T}$.
- For $i \notin 2\mathbb{Z}$, $i \geq 1$, we have that $P_i = x_{i-1}(t_i)$, $V_i = \frac{P_{i-1}V_{i-1}}{P_i}$ and $N_i = N_{i-1}$.

We leave the numerical implementation of the above construction for future work.

Criticisms

- This model considers 2 phases. First we have a "big drop of water" coming out and immediately after it has come out, a bubble of air starts forming. This is not exactly what we observed in the experiments; we will comment on that below, but note that there is not a total axial symmetry.
- There is no mechanism in this model for the upward motion of the liquid-gas interface, which is observed when a bubble of gas enters the bottle.

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11.5 Conclusions

The results of our experiments helped us to understand the dynamics of the exchange between water and air. We observed that on the whole water and air do not share the same space at the same moment. First, water comes out and then air comes in the bottle, never in the same time but with the same duration, so we have a periodic situation. The fact that air and water do not share the same space at the same moment has two important consequences. The first is we must use a specific model for each situation. That's why we use different equations if water comes out or air comes in. The second is that when air comes in the bottle, the total volume increases even if air is very compressible so we must observe periodically the same rising of the level of water corresponding at the volume of air which comes in the bottle.

With the Clanet model, we are able to observe at the short scale, the rising of the level of water due to the air. That's a good point for this model, because we have got a good precision at the short scale, to model the formation of bubbles and to make simulations on a little number of periods. But with this model and with the long scale, we have did such approximations which have lost the periodicity of the model. That's why nothing can make the proof that it's a good model at the long scale even if we are able to choose some parameters to get simulations very close with experimentations. But this model presents the other advantage to be easy to derive and to implement which were very useful to make simulations during the ECMI Modelling Week.

With the Kohira model, we keep the periodicity so that's interesting to look at the long scale of time. The model converges to a periodic solution which represents the characteristic duration between two formation of bubbles. But now, with this model we can't get a good description at the short scale because we can't observe the rising of level of water. So there is a lack of description at the short scale but nothing can make the proof that this lack is a too big error of approximation at the long scale. So if the Clanet model is interesting to describe the dynamics at the short scale of time, the Kohira model is more interesting to describe the dynamics at the long scale. But during the week, we didn't have the time to make simulations.

The future works could be to make simulations with the Kohira model and compare with Clanet model. We could also derive a model which preserves all aspects of that we have observed with the experiments. Another point is to understand if it's a good approximation to consider a uniform level of water in space at the short scale and at the long scale. To study this aspect, an interesting work could be to make simulations (for instance by finite elements on a mesh in space of a bottle) with a two phase model in keeping the Navier-Stokes equations for water and Euler equations for air and to see the differences of the evolution of the level of water with the Clanet model and the Kohira model.

The models that we have considered do not answer a number of questions. For example. it is not clear what actually stops the water to continue flowing out once it starts flowing. It is not clear either (see Figure 2 in [3]) what kind of transition is it from no-flow to oscillatory flow, or from oscillatory flow to fully developed (catastrophic) counterflow. One possibility, for which we have experimental evidence, is that the transition between oscillatory flow and counterflow is actually a gradual one. That is, it is not inconsistent with the experimental evidence that there is *some* counterflow of air during the outflow of water. Such a situation would also explain what is it that stops water from flowing at some stage: the air that enters is trapped at the bottom of the bottle, but when enough of it has entered, it reorganises itself into a spherical cap bubble (such as is considered in [2]), and as it does so, it plugs the opening of the bottle preventing water from flowing out. Then the spherical cap bubble rises, and when it has detached itself from the opening, the water flows again. To see how realistic this alternative mechanical picture is, better experiments are required.

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