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Group 6

Optimization of an antenna network

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Abstract

The task of maximizing the quality of a signal generated by an antenna network on a given territory is quite demanded in industry (e.g. for mobile operators' needs).

In this paper a simple model of antenna network is introduced. Some results with solving optimization problem for finding the antennas' powers are obtained. Further research directions are suggested.

6.1 Introduction

How Antennas Work

Almost every device nowadays has to receive and transmit information for long distances, for example our cellphones, TVs, computers and so on, so every of these devices have to have antennas.

An antenna is an electrical device which converts electric power into electromagnetic waves when transmitting information and vice versa when receiving.

As explained earlier, the electromagnetic waves are created by the interaction between electric and magnetic fields, and consists of a propagation of those fields.

Figure 6.1: Antenna

The antenna could be approximated with a magnetic dipole. A magnetic dipole consists of a pair of magnetic poles (like a magnet) or a closed loop of electric current. The approximation of the current loop is what fits to the antenna model.

When a current flows through a loop it creates a magnetic field. When a loop is in a magnetic field, this field creates a current in the loop.

The other approximation we did was to take the radius of the loop to be small and then the antenna can be expressed as point by $m = \delta_x \mu$.

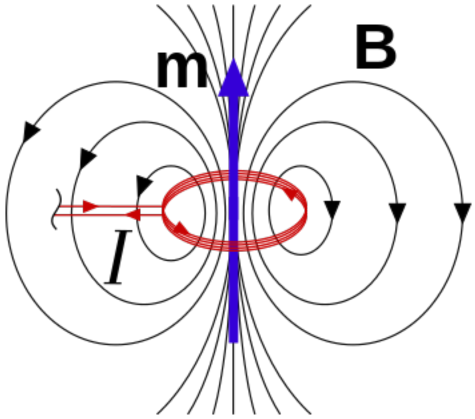

Figure 6.2: Magnetic field

Brief Description of Electromagnetic model

Maxwell's equations [4] for electrodynamics are :

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{6.1}$$

$$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \tag{6.2}$$

$$\nabla \cdot \vec{D} = \rho \tag{6.3}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.4}$$

The electric field intensity $\vec{E}$ is related to electric flux density $\vec{D}$ by $\vec{D} = \varepsilon\vec{E}$ and magnetic flux density $\vec{B}$ is related to magnetic field intensity $\vec{H}$ by $\vec{B} = \mu\vec{H}$, where μ and ε are, respectively, the permeability and the permittivity of the space. Other quantities present in these equations are $\vec{J}$ which is the current density and ρ which is the volume charge density.

The equations of the magnetic field and the electric field shows how the fields are distributed in space, we will consider the static situation for the following explanation.

The divergence operator tell us about the sources of the vector field, for the electric field the divergence is equal to the charge density, this means that the lines of the electric field have an origin at an electric charge. Since the divergence of the magnetic field is zero, it means that there is no "magnetic charges" in the origin of the lines of the field.

The curl equations shows the behavior of the field lines. For the electric field we have zero curl which implies that the lines are radial and for the magnetic field the curl equation means that the lines are closed paths.

For the resolution of the problem we did some approximations of the general model. First we discard the time variation, i.e. we will only work on the static situation. There is no current so $\overrightarrow{J}$ is zero and also there is no charges so ρ is also zero.

The equations then become:

$$\nabla \times \vec{H} = 0 \qquad (6.5)$$

$$\nabla \times \vec{E} = 0 \qquad (6.6)$$

$$\nabla \cdot \vec{D} = 0 \qquad (6.7)$$

$$\nabla \cdot \vec{B} = 0 \qquad (6.8)$$

Some definitions from functional analysis

Let $\Omega \subset \mathbb{R}^3$ is an open set and $D(\Omega, \mathbb{R}^p)$ is infinitely differentiable with a compact support i.e.

$$D(\Omega, \mathbb{R}^p) = C_0^\infty(\Omega, \mathbb{R}^p)$$

Let $D'(\Omega, \mathbb{R}^p)$ is the topological dual of $D(\Omega, \mathbb{R}^p)$, i.e. it is defined as the space of all bounded linear functionals on $D(\Omega, \mathbb{R}^P)$ space.

We will use the following notation

$$u(\varphi) = \langle u, \varphi \rangle_{D'D} \in \mathbb{R},\ \ \forall u \in D'(\Omega; \mathbb{R}^p), \forall \varphi \in D(\Omega; \mathbb{R}^p)$$

Given $u \in D'(\Omega; \mathbb{R}^p)$, we say that $\frac{\partial u}{\partial x_j} \in D'(\Omega; \mathbb{R}^p)$ is partial derivative, if $\forall \phi \in D(\omega, \mathbb{R}^p)$ is true that

$$\left\langle \frac{\partial}{\partial x_j} u, \varphi \right\rangle_{DD'} = -\left\langle u, \frac{\partial}{\partial x_j} \varphi \right\rangle_{D'D}$$

Properties:

- If $D'(\Omega; \mathbb{R}^p) \equiv L^2(\Omega, \mathbb{R}^p)$ and $u \in D'$ then

$$\langle u, \varphi \rangle_{DD} = \int_\Omega u(x)\varphi(x)dx$$

- $D(\Omega, \mathbb{R}^p)$ is dense in $D'(\Omega, \mathbb{R}^p)$.

Example: Let

$$H(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{otherwise} \end{cases}$$

Compute H'. For every $\phi \in C_0^\infty$

$$\int_0^\infty H'(x)\varphi(x)dx = \langle H', \varphi \rangle_{DD'} = \langle H, \varphi' \rangle_{D'D} = -\int_{\mathbb{R}} H(x)\varphi'(x)dx = -\int_0^\infty \varphi'(x)dx = \varphi(0)$$

Therefore

$$\int_0^\infty H'(x)\varphi(x)dx = \varphi(0)$$

So $H'(x)$ is called Dirac delta function [3] i.e.

$$H'(x) = \begin{cases} +\infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

6.2 Methodology

Problem Statement: Given a map Ω, a set of antenna x_i, with $i \in I$, a set of radii r_i which define "exclusion zones" on the map, find the set of $\mu \in r^3$ for each x_i that minimizes the cost of installing the antenna, while maintaining a bounded power across the whole map.

Minimize:

$$c(\mu) = \sum_{i \in I} c_0 + c_i|\mu_i|^2 \tag{6.9}$$

Subject to:

p = f(μ)

$p_{min} \leq p \leq p_{max}$

As indicated by the problem statement, our task requires us to guarantee that the power at any point in the map is with the bounds $[p_{min}, p_{max}]$. As discussed previously, the power is expressed as a PDE as follows:

$$p = f(\mu) \tag{6.10}$$

As an explicit closed-form solution for our PDE is not available, and it is infeasible for us to solve it for a set of infinite points, we rely on finite element methods to provide a numerical approximation given a set of finite points on the map. The Finite Element Method (FEM) is a technique to solve PDEs numerically. It involves generating a polygonal mesh that gives a discrete approximation of a continuous a geometric domain, and evaluating the function in question over this finite set of points.

Let Ω be a bounded domain in R^2 with a polygonal boundary $\partial\Omega$. Ω can be exactly covered by a finite number of triangles, where the intersection of any two triangles is a vertex, a complete edge, or empty.

The boundary of omega is defined by a piecewise parametric equation, where individual edges intersect only at their endpoints. For each edge in the boundary, we define a density of triangles as a basis for triangulation.

Definitions:

A k-simplex is a k-dimensional polytope which is the convex hull of its $k+1$ vertices. In lower dimensions, a 0-simplex is a vertex, a 1-simplex is a line segment, and a 2-simplex is a triangle.

A simplicial complex in in R^d is a set K of simplifies in R^d satisfying the following conditions:

1. Every face of a simplex in K is also a simplex in K
2. If P and P' are both simplifies, then their intersection is a common face of both.

The body $|K|$ of a simplicial complex K is the union of all simplices. When a subset P of R^d is the body of a simplicial complex K, then K is said to be a *triangulation* of P. For a finite set S of points in R^d, a triangulation of S is a simplicial complex K with $|K| = conv(S)$.

The Delaunay triangulation [2] for a set S of n points in a plane is a triangulation $DT(S)$ such that no point in S is inside the circumcircle of any triangle in $DT(S)$.

The Delaunay triangulation of a discrete point set S is the dual of the Voronoi diagram for S, and in applications the Delaunay triangulation of S is computed by constructing the Voronoi diagram of S.

The Voronoi diagram of a set S of n points in a plane consists of N polygons $V(i)$, each centered on point i such that the locus of points on the plane nearest to node i are included in $V(i)$. The Delaunay triangulation is then obtained by connecting the points associated with neighbouring Voronoi polygons, forming a simplicial complex.

The Delaunay-Voronoi algorithm we use attempts to generate a uniform mesh, that is triangles of approximately equal dimension for the given boundary densities. It does this by generating a grid of points S and maximizing the minimum angle of all the angles of the triangles in the triangulation of S, thereby avoiding "skinny" triangles.

We can tune the coarseness of the triangulation with our choice of densities - high densities where field interference is likely to be high, and low densities near the outer boundaries where there may be less interference and thus larger triangles are sufficient to capture the power of a region.

Notice that the definitions we have included this far relate to generating a mesh for polygons (or polyhedra). In our case, the exclusion zones are defined by circles rather than polygons. Thus we approximate the boundary of the circle using line segments. The more dense the mesh around these curved boundaries, the closer we approximate the curvature and thus the closer we get to a globally optimal solution. However, there is a trade-off; the finer the mesh, the more computationally expensive the subsequent optimization.

Thus, our procedure for approximating power using FEM is as follows:

Step 1: Define the map. Given a test area, fixed antenna points x_i, and a set of radii r_i with $i \in I$, we generate a map for computing the field, including both internal and external boundaries. For example, consider the map below for a set of 4 antenna in R^2 (see Figure 6.3).

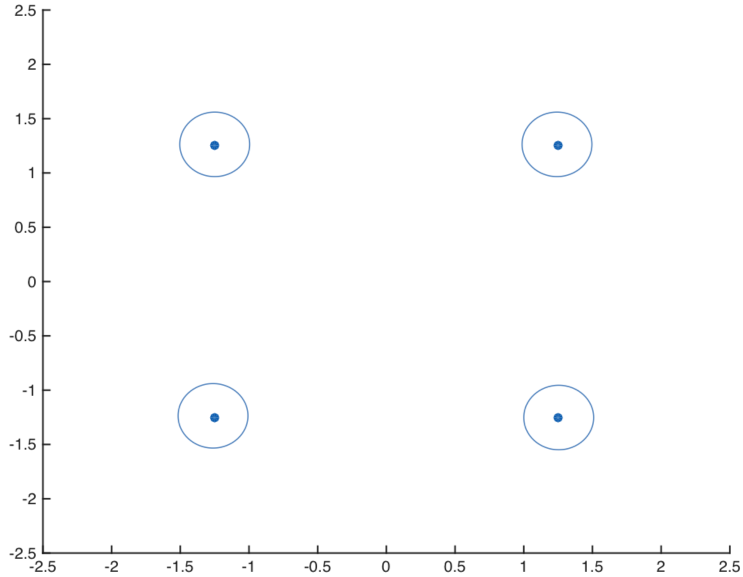

Figure 6.3: Map

Step 2: Generate a mesh. Given the map, the Delaunay-Voronoi algorithm generates a mesh given the number of triangles we assign to each piece of the boundary. The density of inner vertices in the triangulation is determined by the density of the points on the boundary (see Figure 6.4).

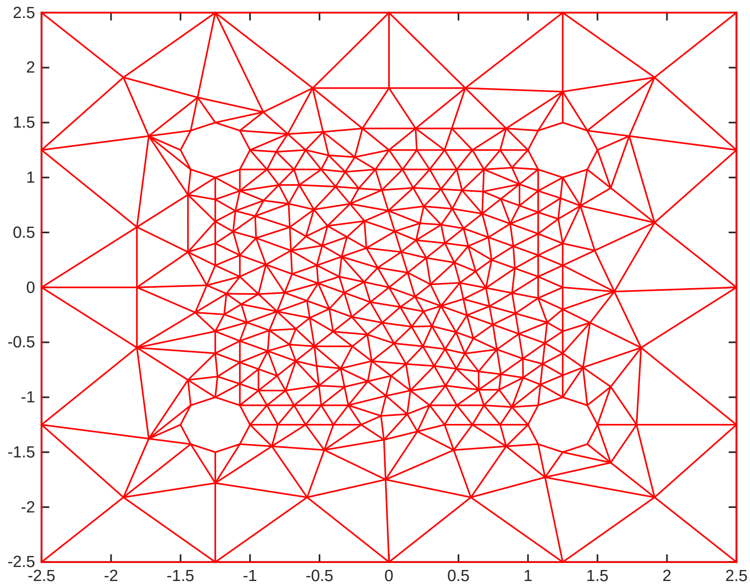

Figure 6.4: Mesh

Step 3: Evaluate the PDE. Within the context of the optimization procedure described in the subsequent section, we evaluate the PDE on the finite set of centers of triangles composing the mesh to approximate the power levels across the entire map.

After the approximation is made, we need to solve constrained optimization problem (6.9). We have convex minimization function and there is a lot of methods of constrained optimization for such functions. Some of these methods converge well when initial solution is quite close to the optimal one and some doesn't. In this kind of problem it is not easy to find initial constrained solution, especially in the case, when antennas are stated on the ground. This happens because of Dirac's mass singularity around the each antenna. So, we need to use an optimization method, which converges well even when the initial solution is not constrained. That's why we selected the exterior penalty function method.

The main idea of this method is to replace solving the constrained problem with similar unconstrained one. Here we describe the procedure of exterior penalty function method.

Step 0: Set $\lambda = \lambda_0$, $t > 1$, ϵ - parameters of the method.

Step 1: Solve unconstrained problem:

Minimize

$$c(\mu) + \lambda g(\mu), \tag{6.11}$$

where $g(\mu)$ is a penalty function. We will describe below how to define the penalty function.

Step 2: Check condition $\lambda g(x^*) < \epsilon$, where x^* is the solution of the unconstrained problem in Step 1. If condition is satisfied, algorithm stops. Otherwise, $\lambda := \lambda_0 t$ and algorithm goes to step 1.

First of all, we selected penalty function this way:

$$g(\mu) = \sum_j max(0, p(x_j, \mu) - p_{max}(\mu))^2 + max(0, p_{min}(\mu) - p(x_j, \mu))^2, \tag{6.12}$$

where x_j are points from the mesh. In general, penalty function has to be zero on constrained solution and non-negative otherwise. As you can see, the function in (6.12) satisfies our needs.

Also we can define penalty function like this:

$$g(\mu) = \sum_j max(0, (p(x_j, \mu) - p_{max}(\mu))(p_{min}(\mu) - p(x_j, \mu)))^2, \tag{6.13}$$

We tried to implement both of variants for penalty function and each of them gave us equal solution.

Then, parameters of the method have the following semantics: ϵ is a tolerance parameter and $t > 1$ is a parameter which describes the 'speed' of convergence of the method. With bigger t method converges faster if it converges but there is a risk to loose convergence. This happens because with the bigger value of λ any method of solving the unconstrained optimization problem doesn't converge well.

At last, we needed to specify an optimization method to solve unconstrained problem. We decided to use Powell's method [1]. This method is a modification of conjugated vectors method, which is well used in problems with quadratic minimization function. We have such function in our unconstrained problem, so that is the reason to select Powell's method. In Powell's method we needed to solve some one-dimensional minimization problems. It was done by using golden division method.

6.3 Results

The first task we had was to compute the field generated by one antenna on each point of the domain (in the magnetostatic approximation). Since we have no current, the equation 6.1 becomes:

$$\nabla \times \vec{H} = 0 \tag{6.14}$$

The equation 6.4 can be written as $\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$ by the following relation: $\vec{B} = \mu_0(\vec{H} + \vec{M})$ where μ_0 is the permeability of free space and $\vec{M}$ is the magnetization vector.

Taking the approximation of a magnetic dipole for the antenna we have $m = \delta_x \mu \ \mu, \subset \mathbb{R}^3$

Since the curl of the vector field $\vec{H}$ is zero, then there exists a scalar-valued function so that $\vec{H} = \nabla f$.

$\Delta f = -\nabla \cdot m$

the solution is given by

$f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{-\nabla \cdot m}{r} \, dy = -\frac{1}{4\pi} \nabla \cdot m \star \frac{1}{r}$

Since h=∇f we get

$h_i(x) = -\frac{1}{4\pi} \nabla_x (\nabla_x \cdot m \star \frac{1}{r})$

$= -\frac{1}{4\pi} m \star \nabla_x (\nabla_x \cdot \frac{1}{r}) = -\frac{1}{4\pi} \delta_x \star \nabla_x (\nabla_x \cdot \frac{\mu}{r})$

the operator $\nabla_x \nabla_x \cdot$ is equal to the matrix:

$\{\frac{\partial^2}{\partial x_i \partial x_j}\}$

Since $r = \sqrt{\sum_{i=1}^{3} x_i^2}$.

h is obtained by $\mu \cdot matrix$

We obtained the next image (see Fig. 6.5) which is very similar to the one in Fig. 6.6 which represents the magnetic flux lines of a magnetic dipole, confirming that our calculations were right.

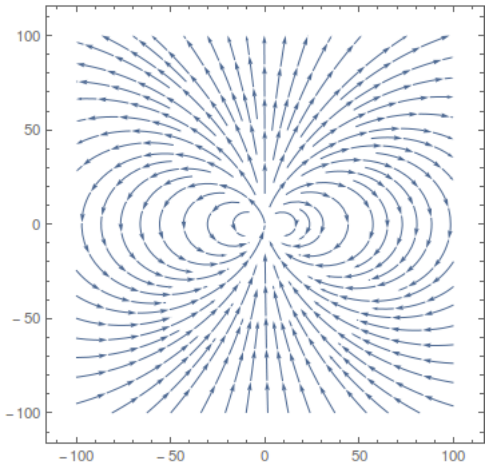

Figure 6.5: Magnetic flux lines in 2-D, with the antenna in point (0,0)

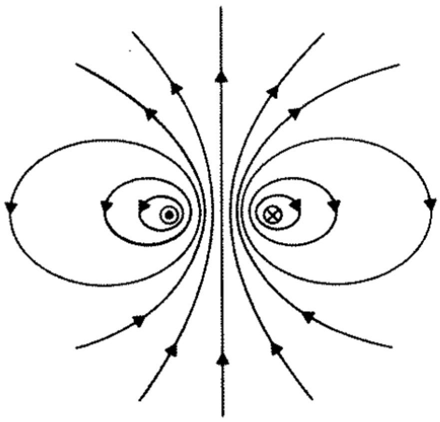

Figure 6.6: Magnetic flux lines of a magnetic dipole

Here we list all properties of antenna network model that we have.

Flat teritory

All the points is in a plain i.e. $(x, y) \in \Omega$, $\Omega \subset \mathbb{R}^2$.

Parameters

1. the magnetic moments

2. constraints

   - The protection zone around the antenna depends on η. For all $x \in \bigcup_{i \in I} B(x_i, \eta) := \Sigma_e x$, we don't do any computations

- minimal power of each point

$$p_{min} \in L^\infty(\Omega; \mathbb{R}^+)$$

- maximal power of each point

$$p_{max} \in L^\infty(\Omega; \mathbb{R}^+)$$

Field and power

$$\Sigma_x h(x) = \sum_{i \in I} h_i(x), \ \ \forall x \in \Omega.$$

Let $p(x) = |h(x)|$ and $\overline{p(x)} = \frac{1}{|w|} \int_w p(x) dx$.

We did numerical calculations with following values of parameters:

1. $\Omega = [-2.5; 2.5] \times [-2.5; 2.5]$ (see Fig. 6.3)

2. $p_{min} = 1$

3. $p_{max} = 100$

We didn't have time to experiment with minimization function (6.9) so we made the cooefficients before $|\mu_i|^2$ all equal to 1.

We approximated power in a triangle with the most simple approximation - with the value of power in the center of the triangle ($\overline{p(x)} = p(x_j)$).

We tried to experiment with $p(x)$. It is possible to say that $p(x) = |h(x)|^2$ because this equation makes sense in terms of energy. However, with this definition of power we got very slow convergence. This happened because with reducing constrained problem to unconstrained one minimization function became not quadratic, which slows convergence for selected Powell's method of solving the unconstrained problem.

So, we did calculations for two scenarios. In the first one antennas were set on the ground, in the second one all antennas had equal nonzero height.

1. Antennas are on the ground ($h = 0$)

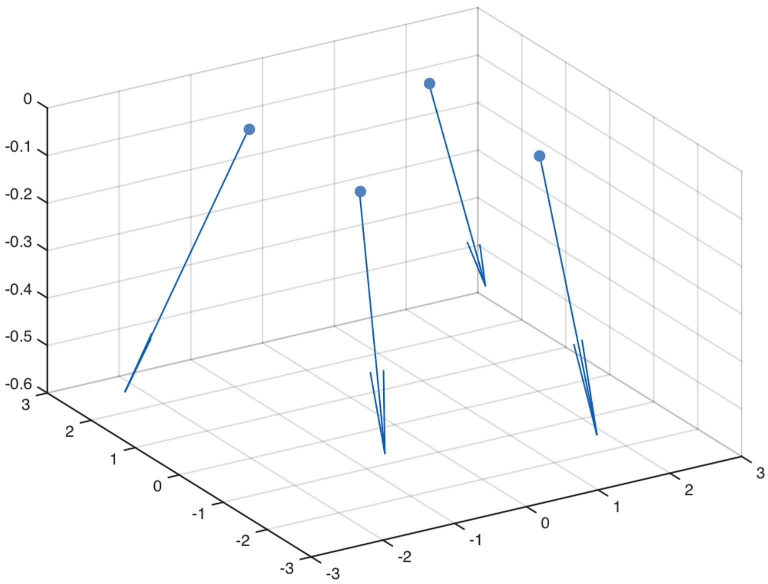

Figure 6.7: Magnetic moments

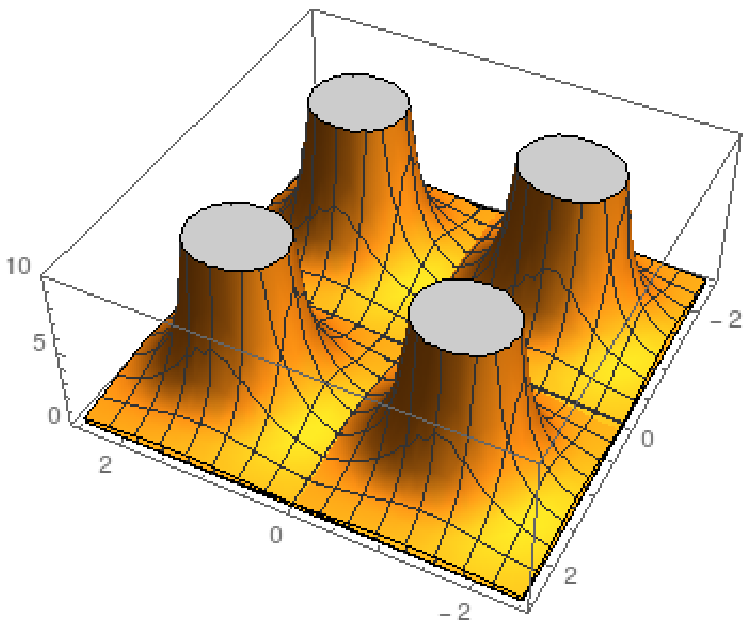

Figure 6.8: Power in the area

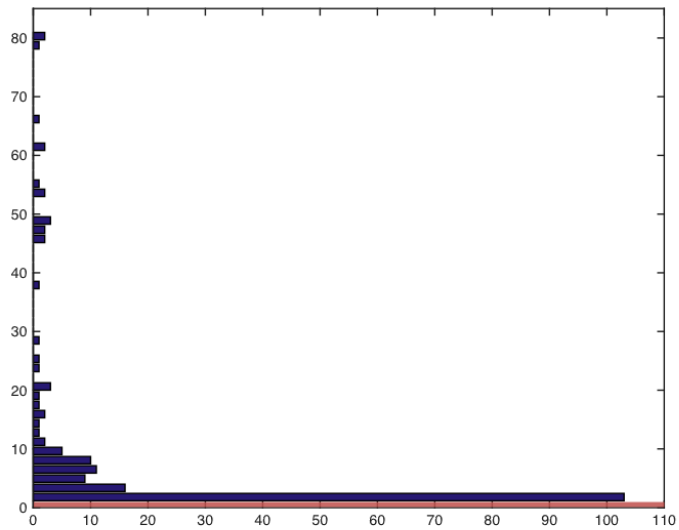

Figure 6.9: Power distribution

2. Antennas are raised off the ground ($h = 1$)

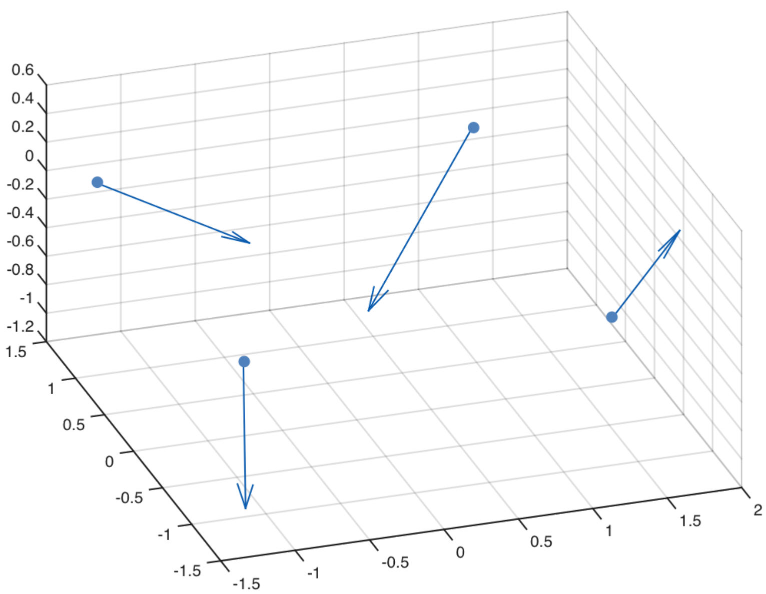

Figure 6.10: Magnetic moments

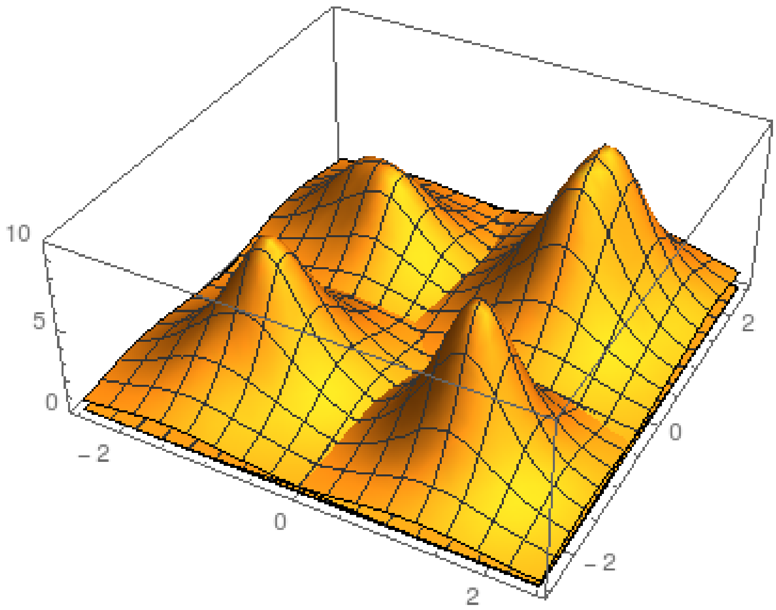

Figure 6.11: Power in the area

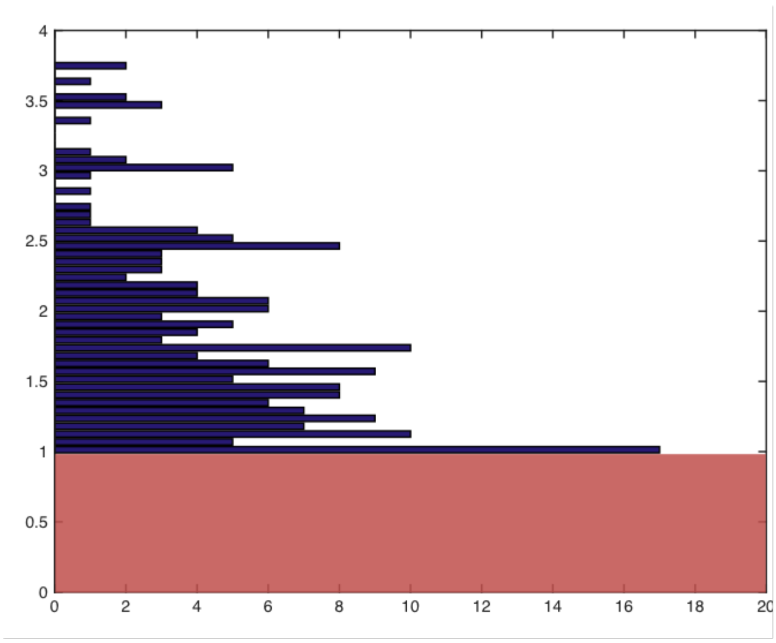

Figure 6.12: Power distribution

We did some observations from the obtained solutions:

- High power levels occur near boundaries of exclusion zones as it has to be.

- Power levels cluster near p_{min} which is a sign of the optimal solution.
- Power is more uniform, when antennas are raised off the ground. It happens because in this case singularity of Dirac's mass doesn't affect field on the ground.
- The maximum level of power is much bigger in the first case because of Dirac's mass singularity.

We also tried numerical experiments with different values of parameters but from all of them we got the same observations.

6.4 Extensions

The all work we did could be extended to some directions.

First of all, we can investigate performance for a more limiting set of constraints because we made some not realistic assumptions because of lack of time. For example, radii of exclusions zone or cost coefficients can possibly change from antenna to antenna. Also we set p_{min} and p_{max} constant in the area but they can change from point to point.

Then, we can set antennas heights and positions as variables and find them by solving an optimization problem. For example, it is required when you need to cover area in the there is no antenna installed (e.g. in country-side).

Finally, the first assumption we made was that wavelengths are big enough and field doesn't vary in time. So, we can think about implementing the generalization where complete Maxwell's equations are solved.

6.5 Conclusion

- The simplified model of an antenna is defined and Maxwell's Equations were applied to this model.
- We created a network of antennae defined on a map and discretized the domain through triangulation.
- We optimized installation cost as a function of antenna properties, with constraints, for multiple cases.
- These results can be extended to more complicated situations.

Bibliography

[1] Powell, MJD. *An efficient method for finding the minimum of a function of several variables without calculating derivatives*, *Computer Journal*, volume 7 (2): 155–162. 1964.

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[3] ———. Dirac delta function. https://en.wikipedia.org/wiki/Dirac_delta_function.

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