Mechanised Justification in "Systems that Explain Themselves" for Mathematics Education

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eduTPS: Working Group on Justification in Doing Math
at CADGME, Coimbra, Portugal
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Outline

"Systems that Explain Themselves"?

- Systems for proofs: well-known theorem provers (TP), e.g. Coq, PVS, HOL, Isabelle have
  - math knowledge deduced from first principles (axioms)
  - so, elements of math **knowledge** "explain themselves"
  - **usage** for proof does **not** explain itself

  $\longrightarrow$ short demo of Isabelle
- Systems for engineering mathematics: only prototypes, e.g. 4ferries (by R.J. Back), Mathtoys, $\mathcal{ISAC}$
  - need to build upon TPs (justifications !)
  - need to step-wise construct problem solutions
  - need to support modularisation into sub-problems
  - . . . such that **usage** and **knowledge** "explain themselves"

  $\longrightarrow$ Isabelle/$\mathcal{ISAC}$ will serve as example

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Term Language

The demo has shown . . .

- principal benefits
  - uniformity over all domains of mathematics
  - type system efficiently excludes ambiguities
  - clear description of functions and respective rules
- added value of implementation
  - a formula's elements are connected with definitions
  - types are transparent by mouse pointer
  - feedback to input of formulas
  - structure of formulas, i.e. sub-terms are transparent
  - internal representation adaptable to engineers' needs

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Proof Language . . .

. . . adapts to conventions of engineering mathematics:

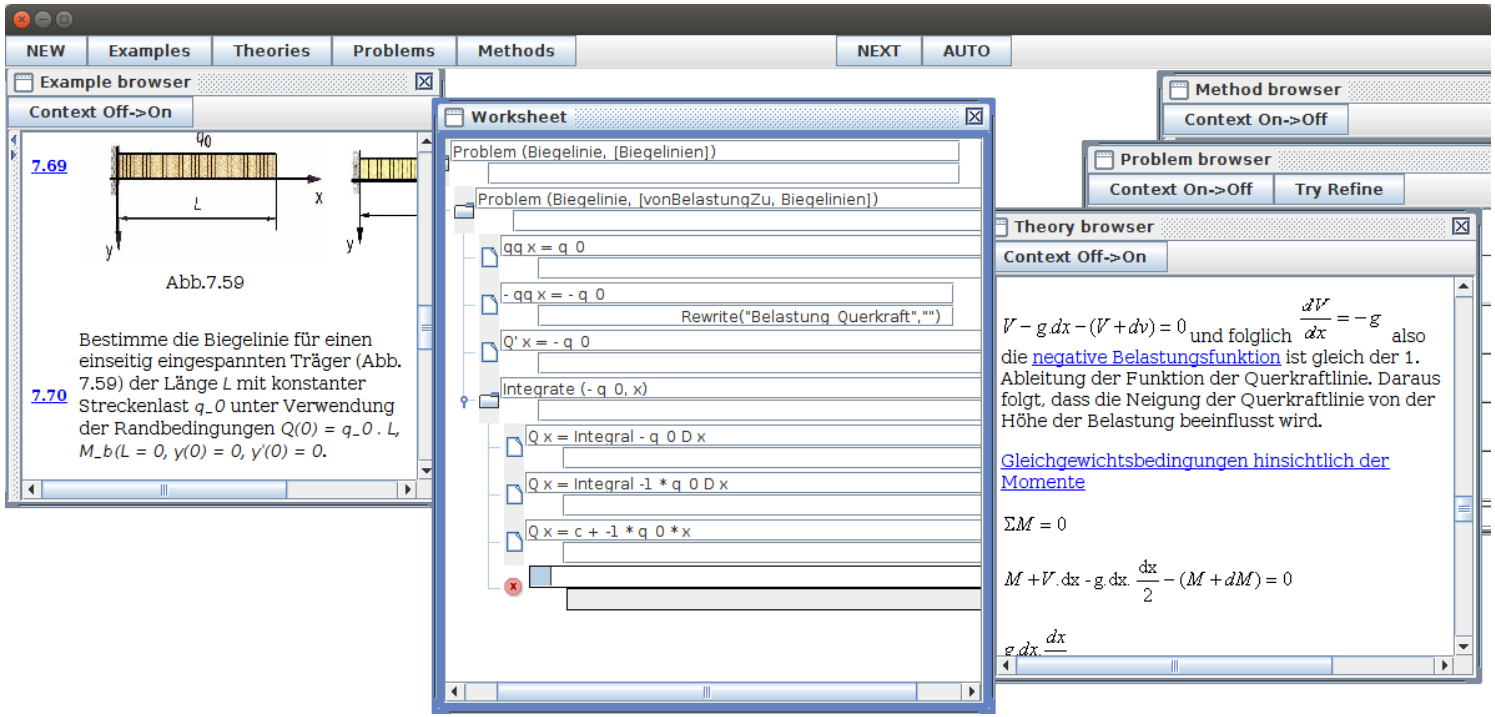

Figure: Conventional worksheet on $\mathcal{ISAC}$'s front-end

Proof Language

- Principal benefits
  - calculations in a conventional format
  - all steps of calculation in a consistent framework
  - each step is justified by theorems
  - specific steps equivalent to Computer Algebra
  - Computer Algebra decomposed into elementary steps
- Added value of implementation
  - change from survey to detail in the calculation tree (collapsing and expanding)
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  - steps can be redone while trying alternative ways
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Specification Language

Formal specification of the previous **Solution:**

```
01 Problem (Biegelinie, [Biegelinien])
02   Specification:
03     Model:
04       Given     : Traegerlaenge L, Streckenlast q_0
05       Where     : q_0 ist_integrierbar_auf [0, L]
06       Find      : Biegelinie y
07       Relate    : Randbedingungen [Q 0 = q_0 · L, M_b L = 0, y 0 = 0, d/dx y 0 = 0]
08     References:
09       Theory    : Biegelinie
10     x Problem   : ["Biegelinien"]
11     o Method    : ["IntegrierenUndKonstanteBestimmen2"]
12   Solution:
```

Hidden data for *“next step guidance”*:

[([*Traegerlaenge L, Streckenlast* q_0*, Biegelinie y,*
Randbedingungen [$Q\,0 = q_0 \cdot L$, $M_b\,L = 0$, $y\,0 = 0$, $\frac{d}{dx}\,y\,0 = 0$]*, FunktionsVariable x*]
(*"Biegelinie",* [*"Biegelinien"*]*,* [*"IntegrierenUndKonstanteBestimmen2"*]))]

Specification Language

- Principal benefits
  - formal specification prepares mechanical solution
  - pre-condition determines solvability
  - post-condition makes essence of a problem explicit
  - problems decomposed into sub-problems (with specifications)
- Added value of implementation
  - specifications can be easily searched and tried
  - trees of specifications allow automated refinement
  - successfully specified problems solved by key stroke
  - sub-problems can be interactively arranged
  - specifications as black boxes raises abstraction in problem solving, see slide movie

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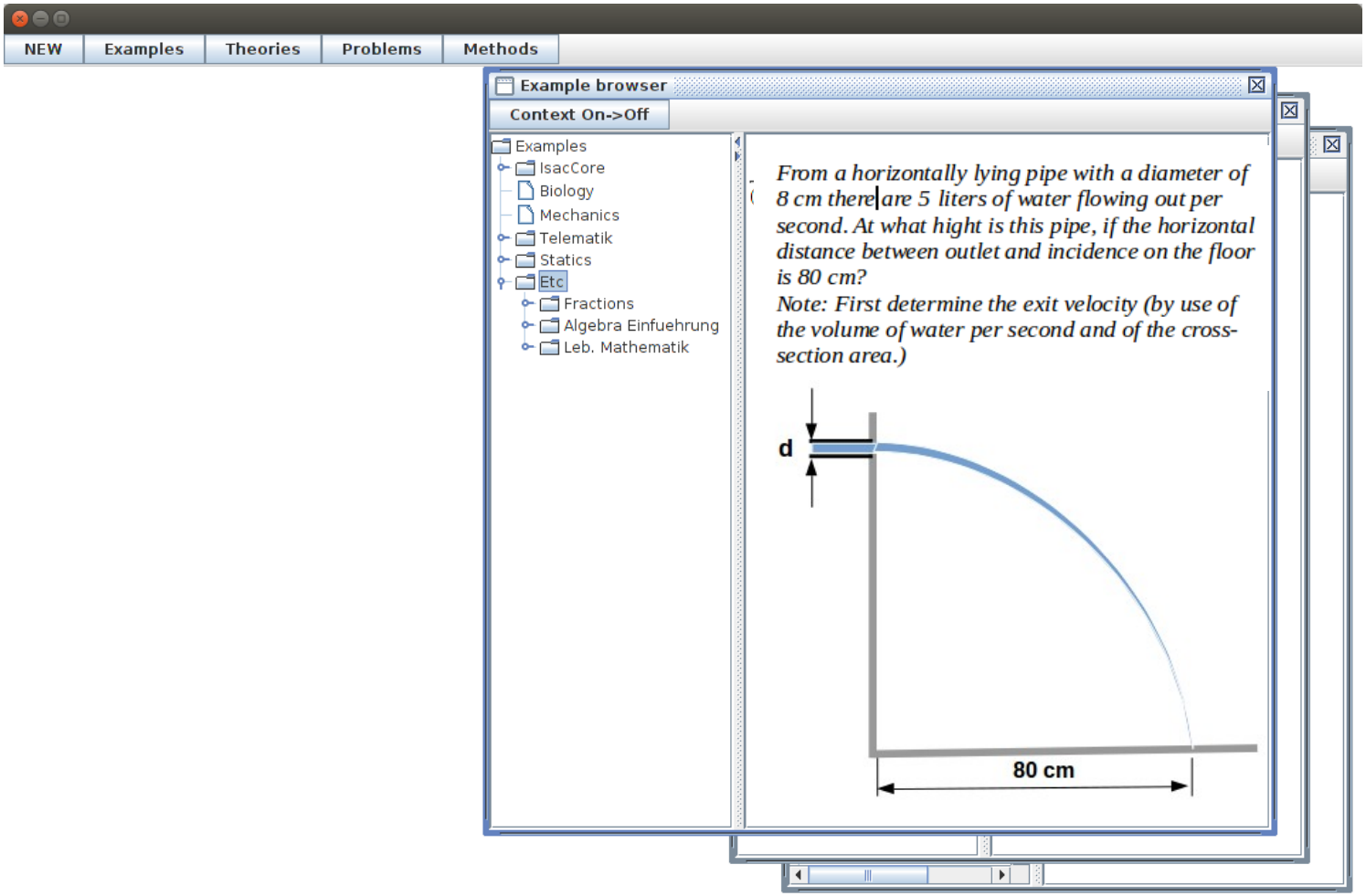
Start Example



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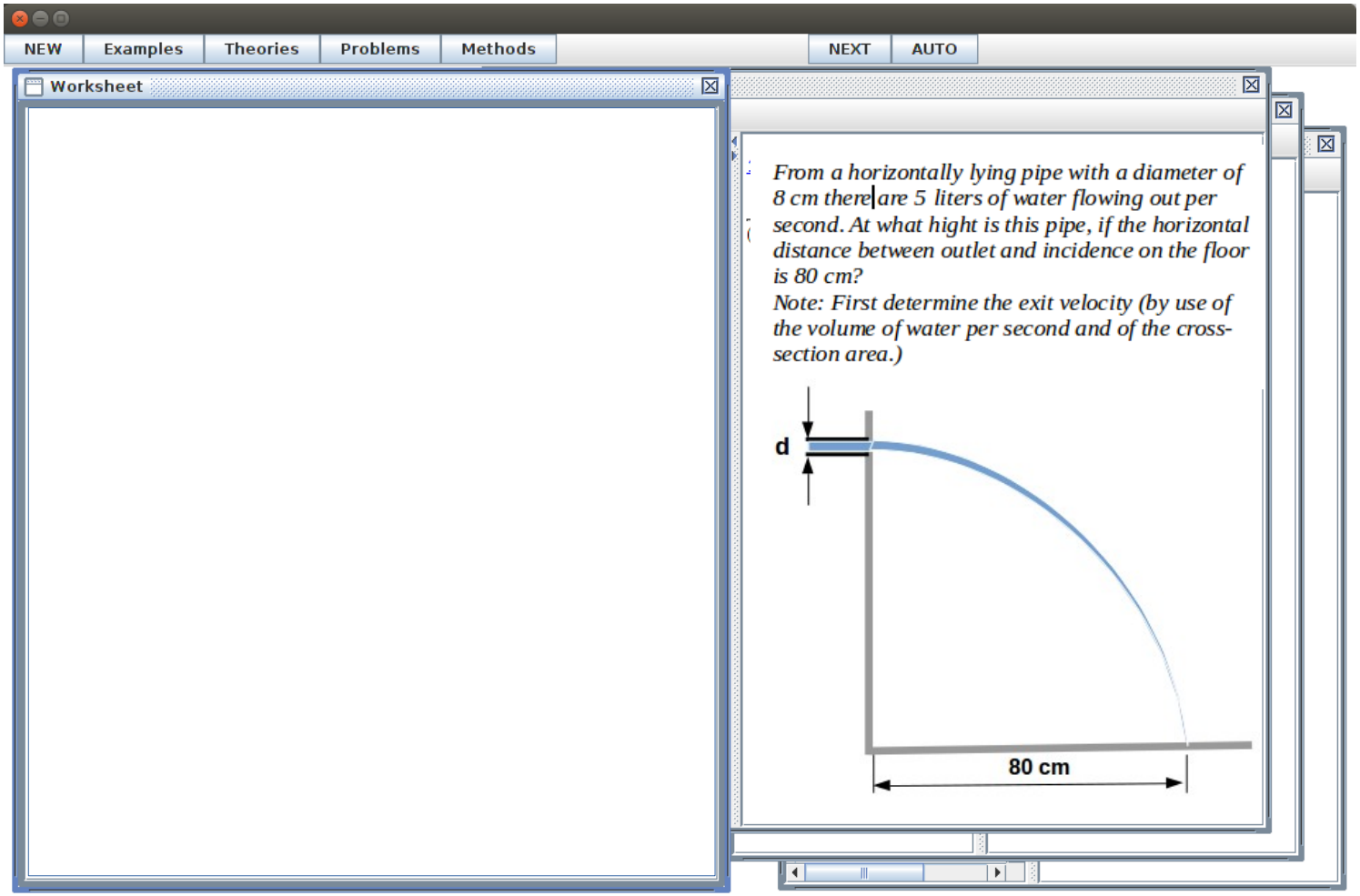
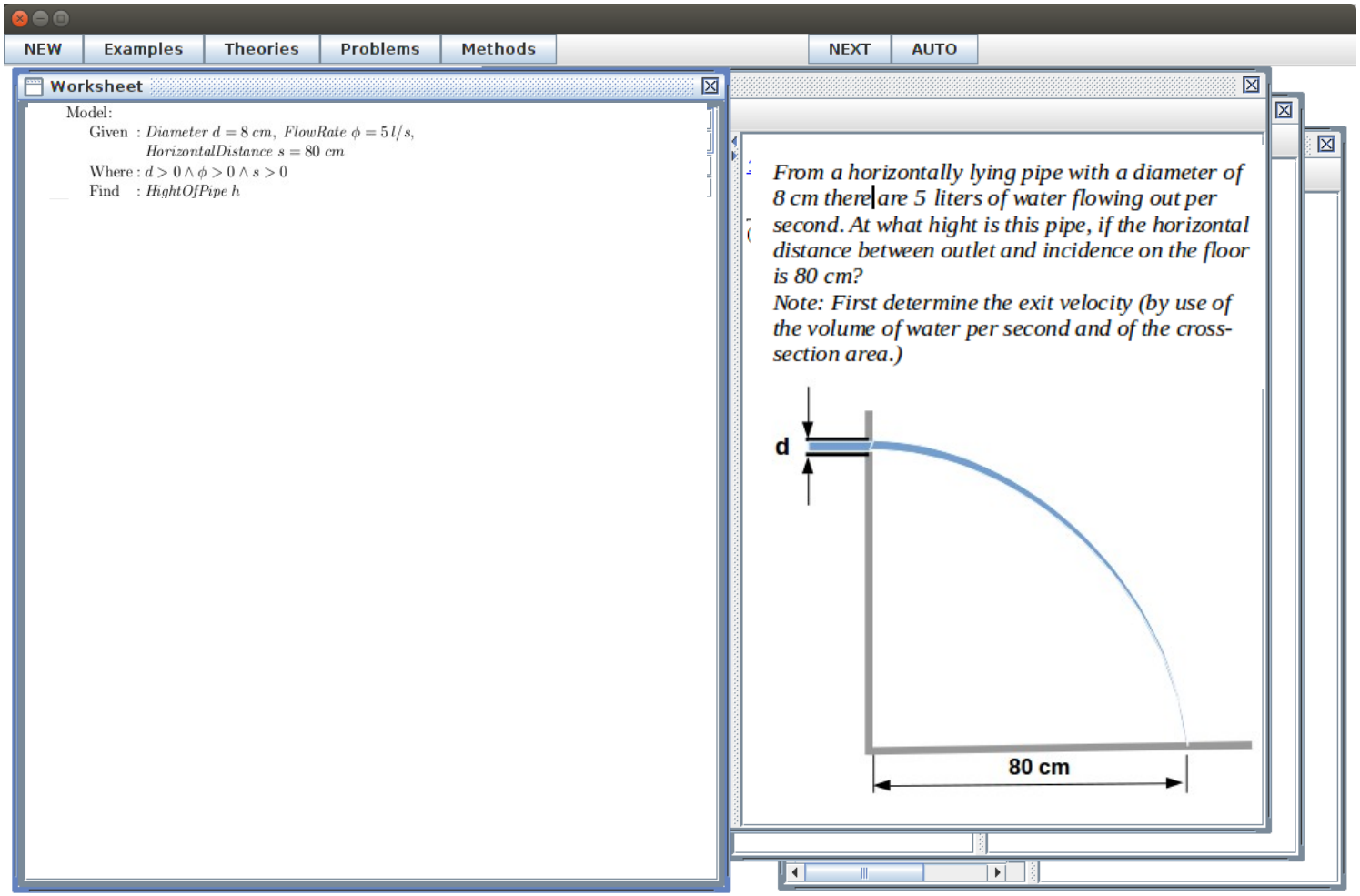
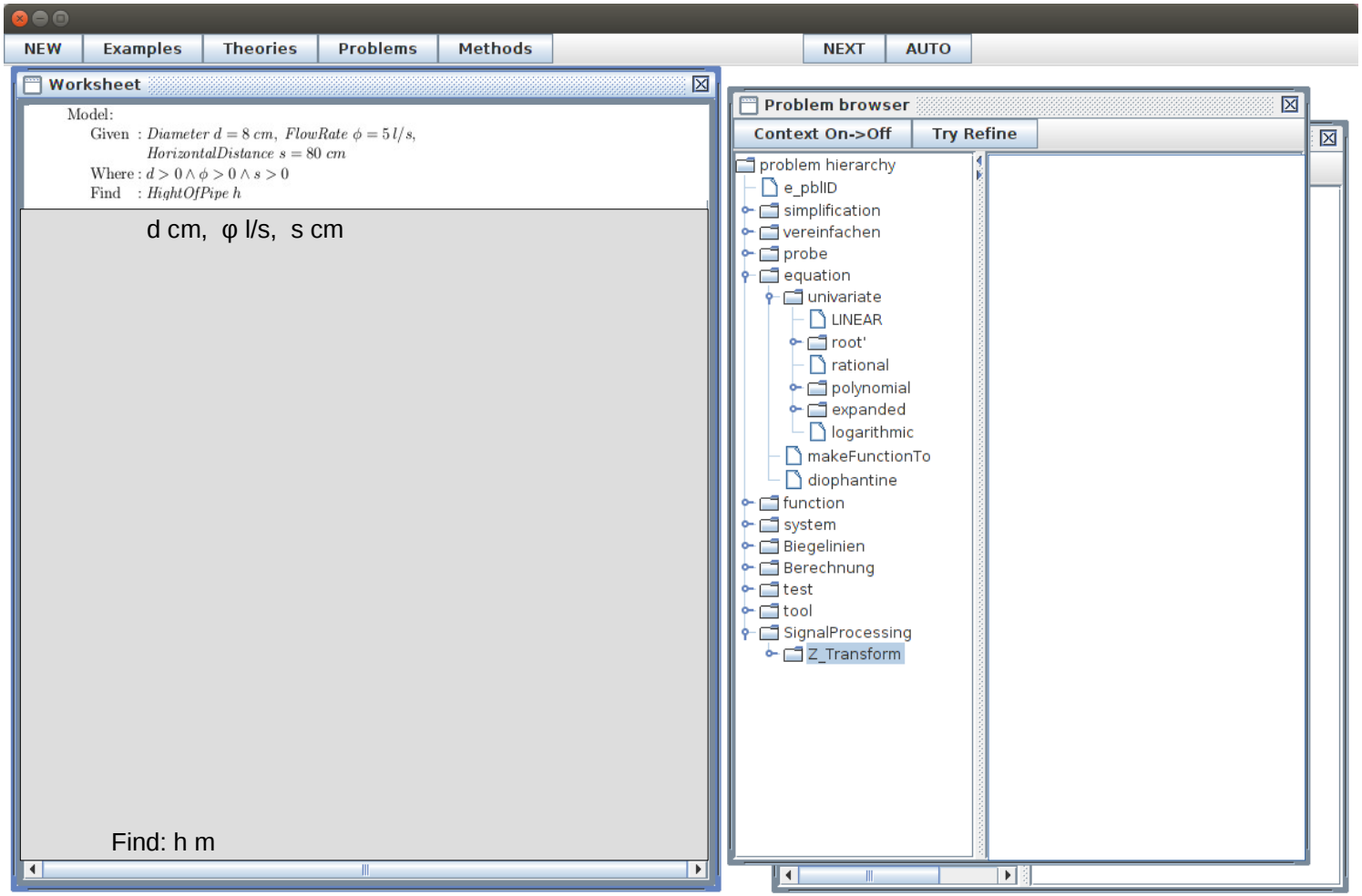


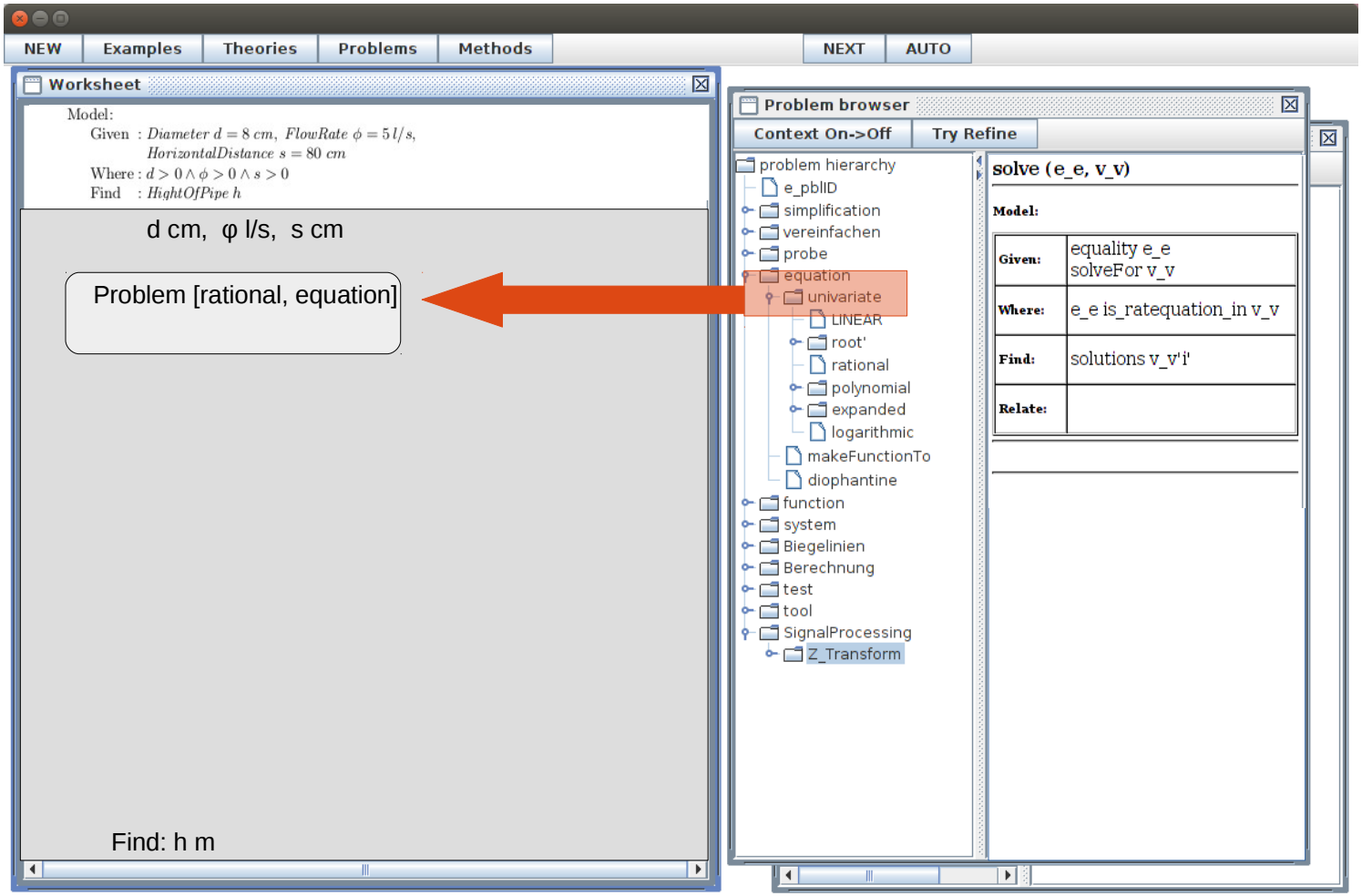
Problem modelled ok



Start considering sub-problems



Select sub-problems



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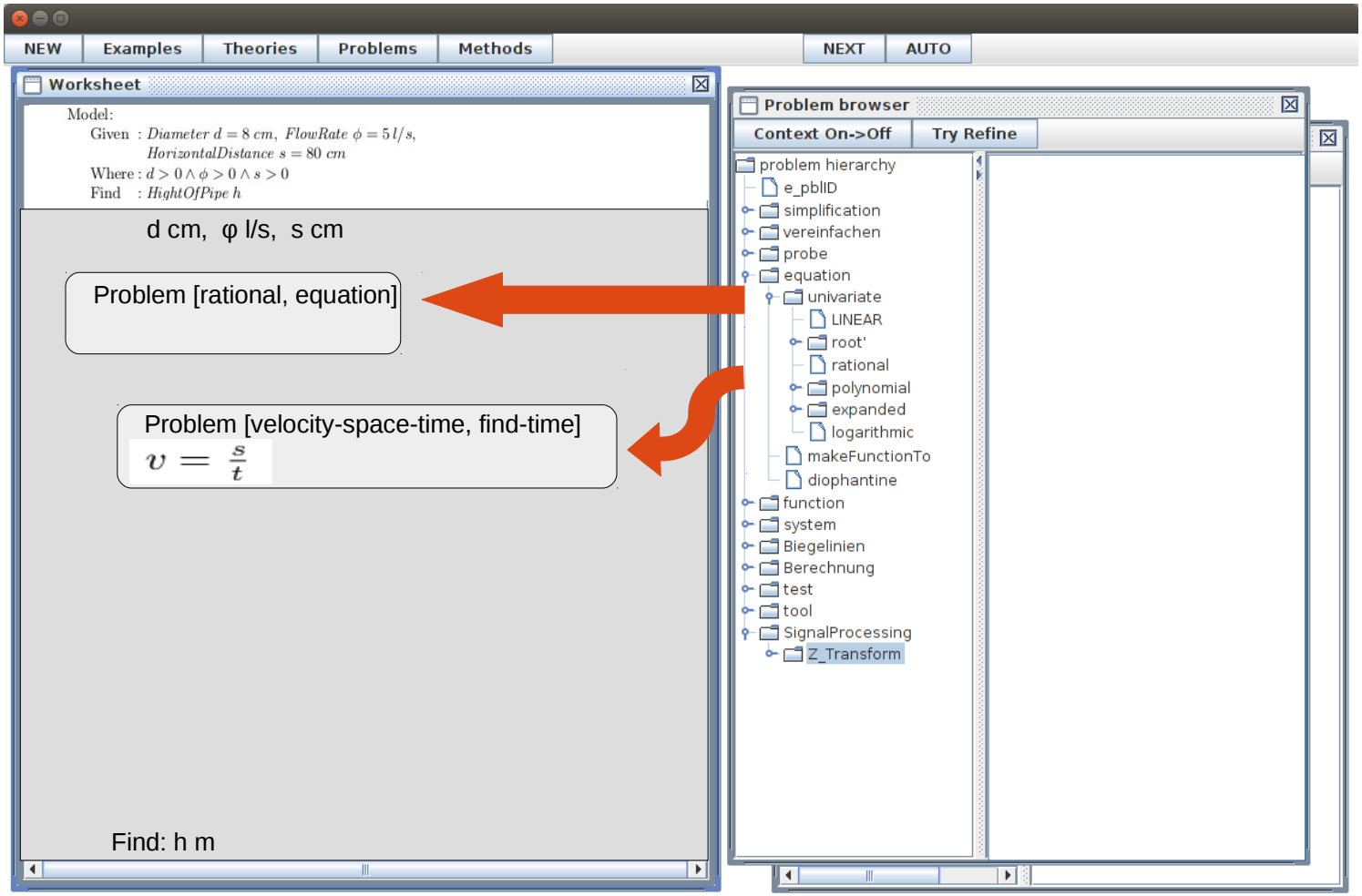

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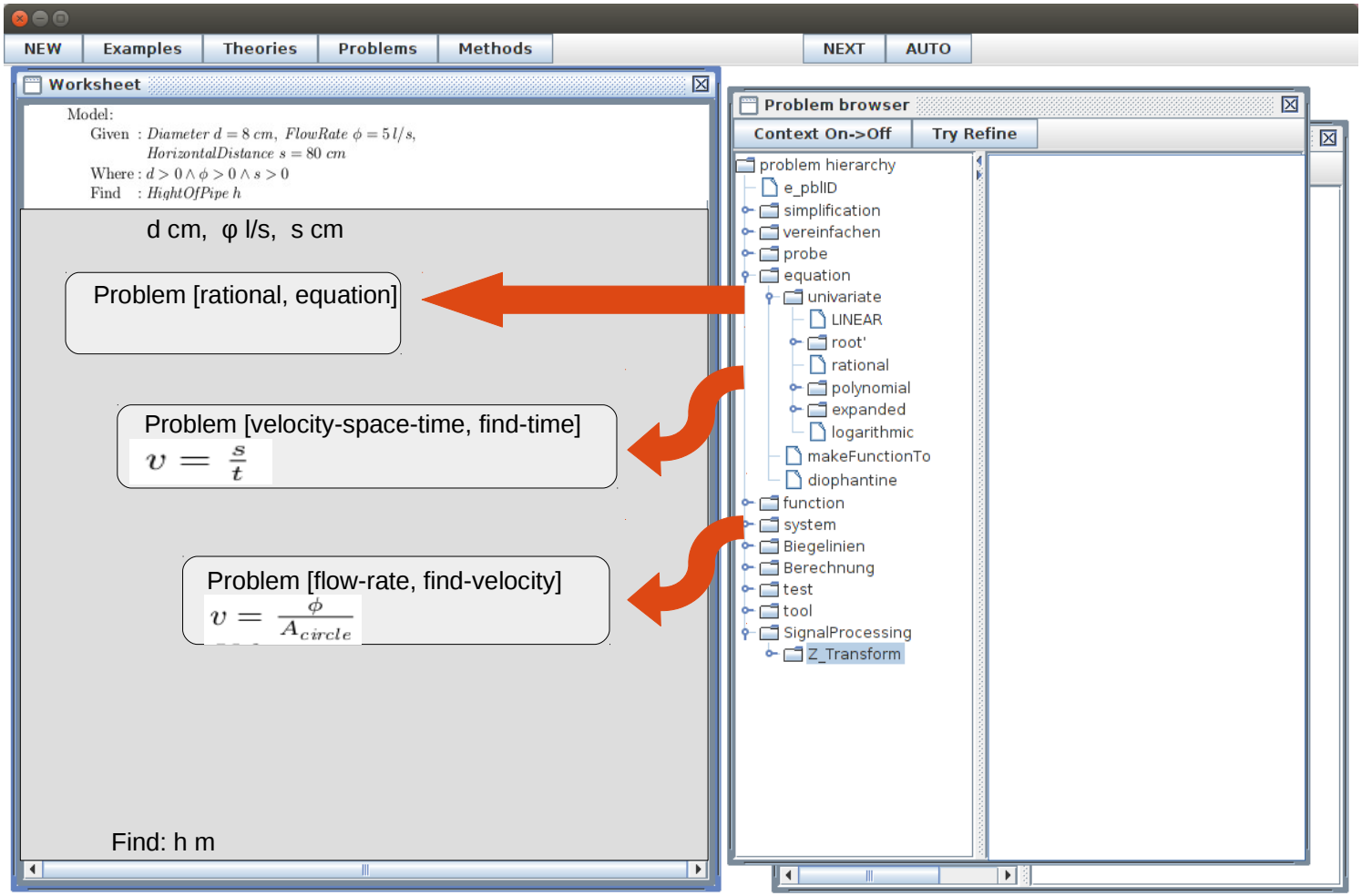

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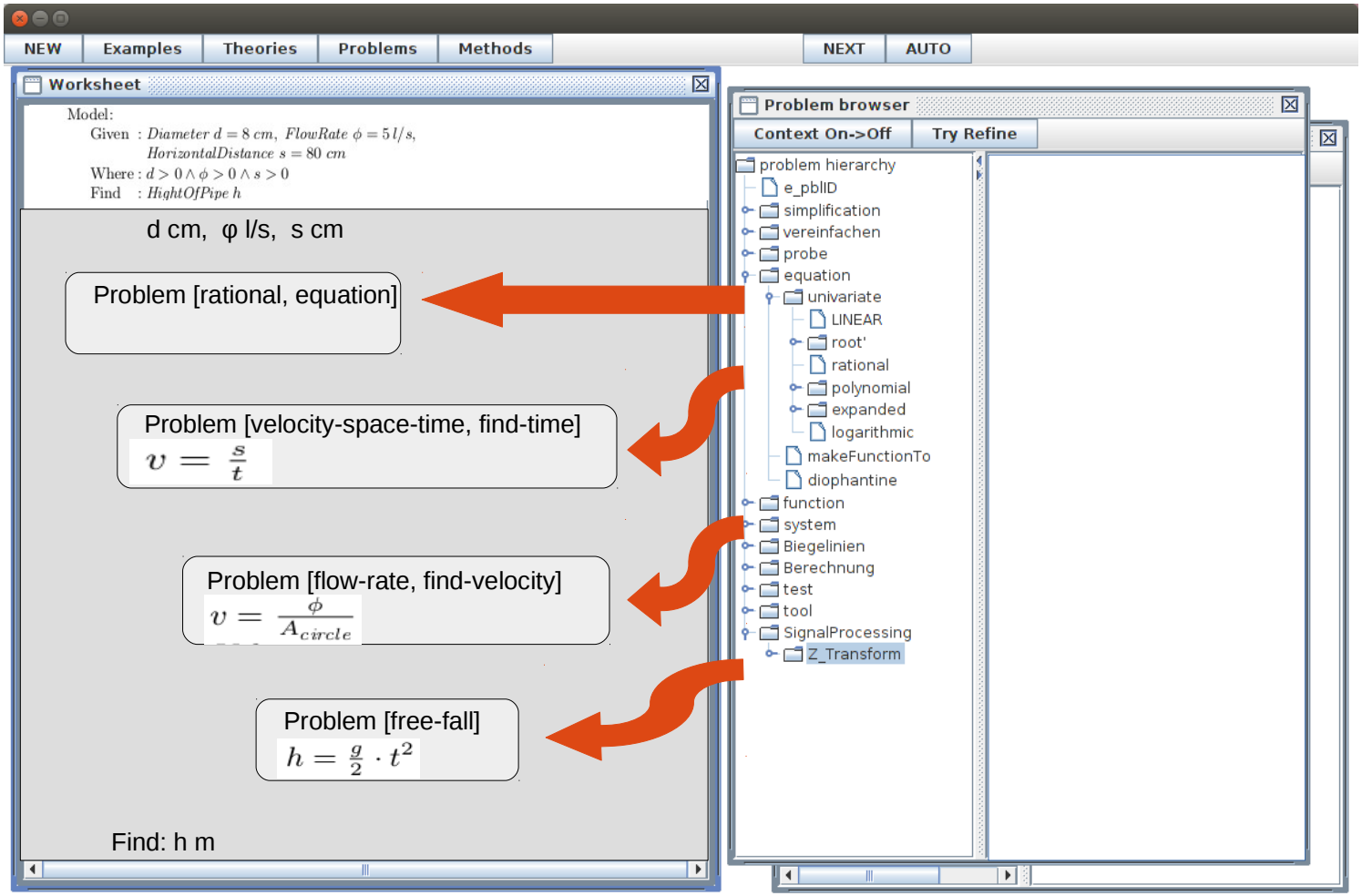
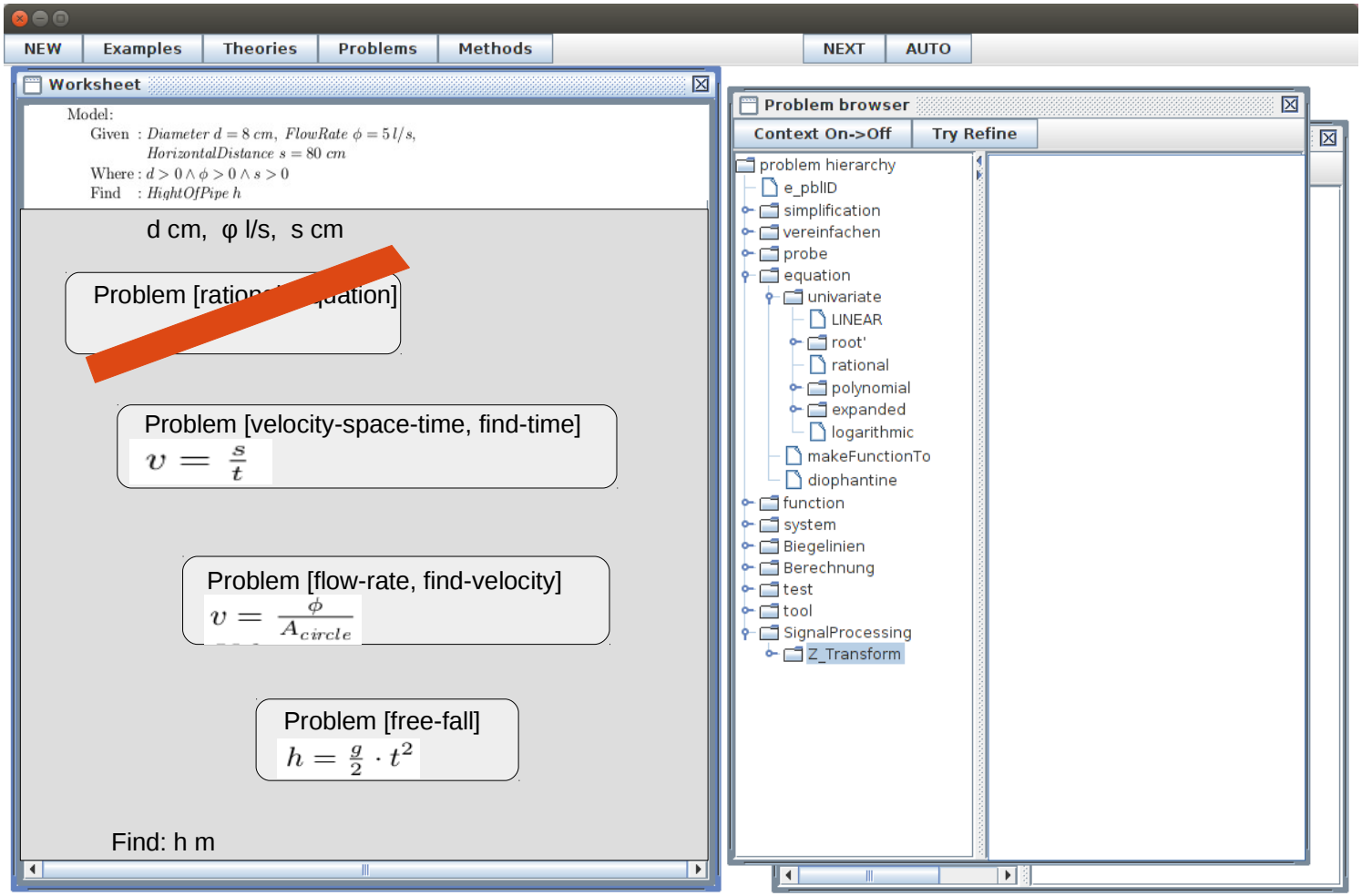


Delete irrelevant sub-problems



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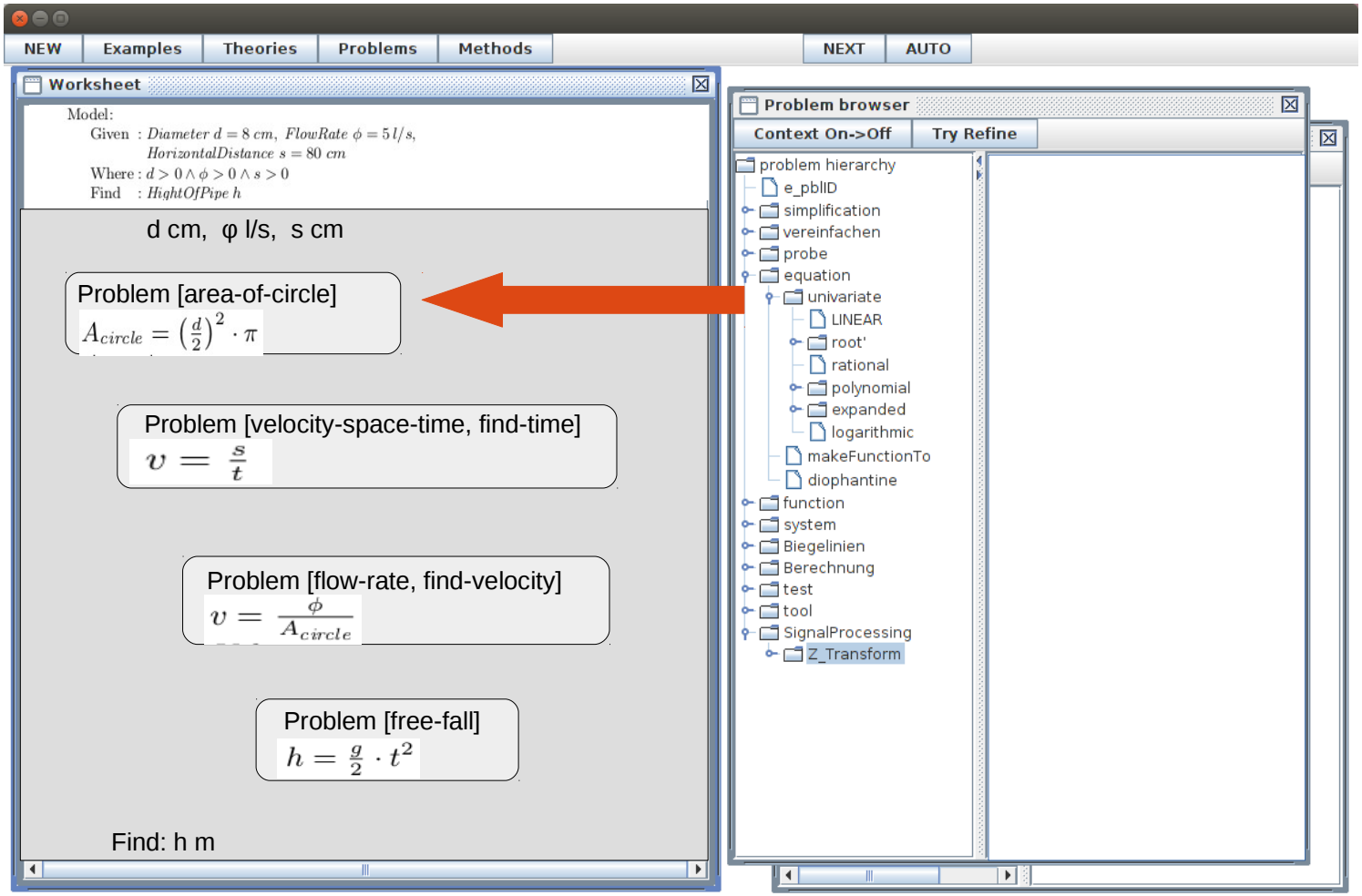
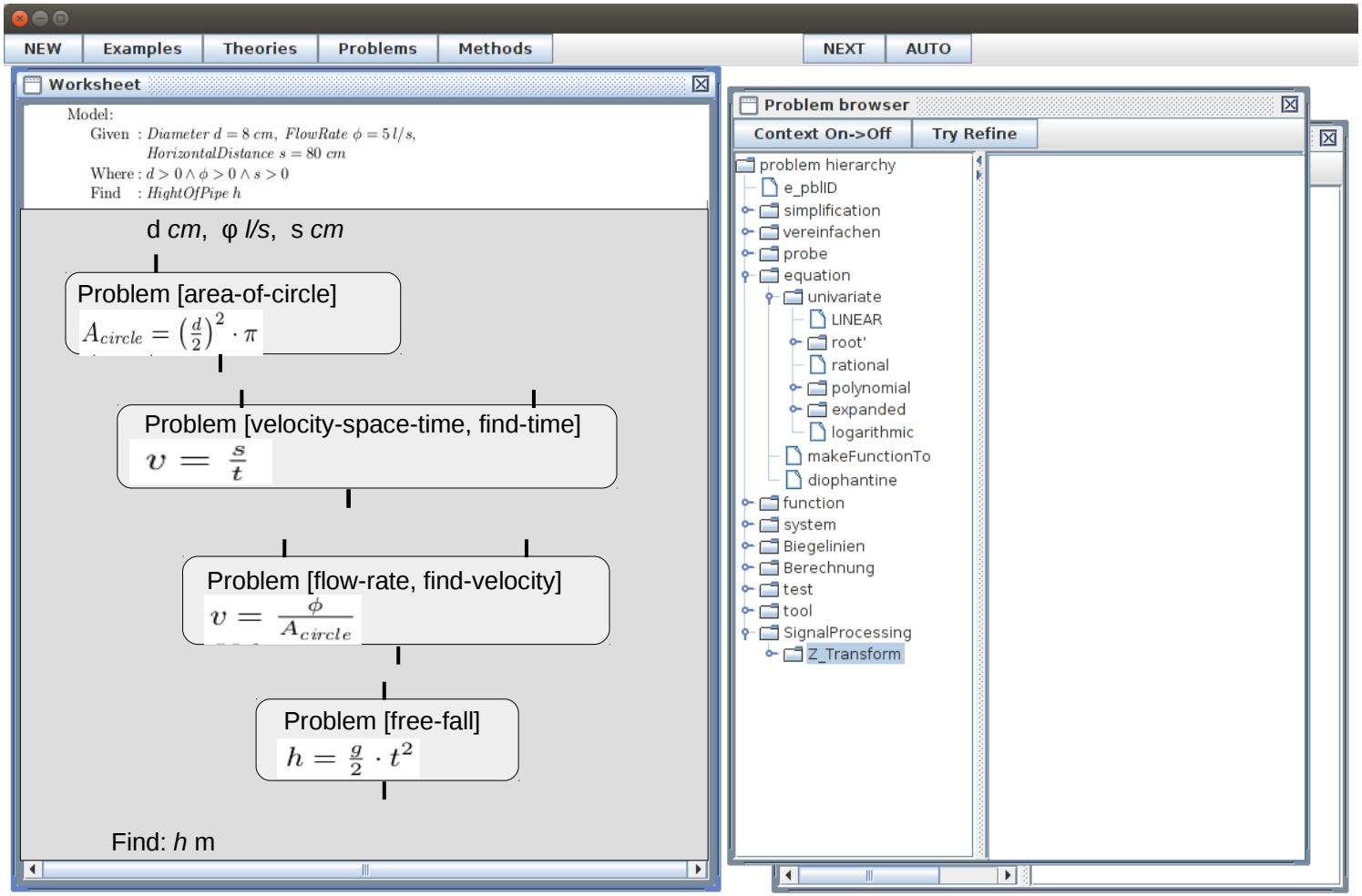


What is given / has to be found?



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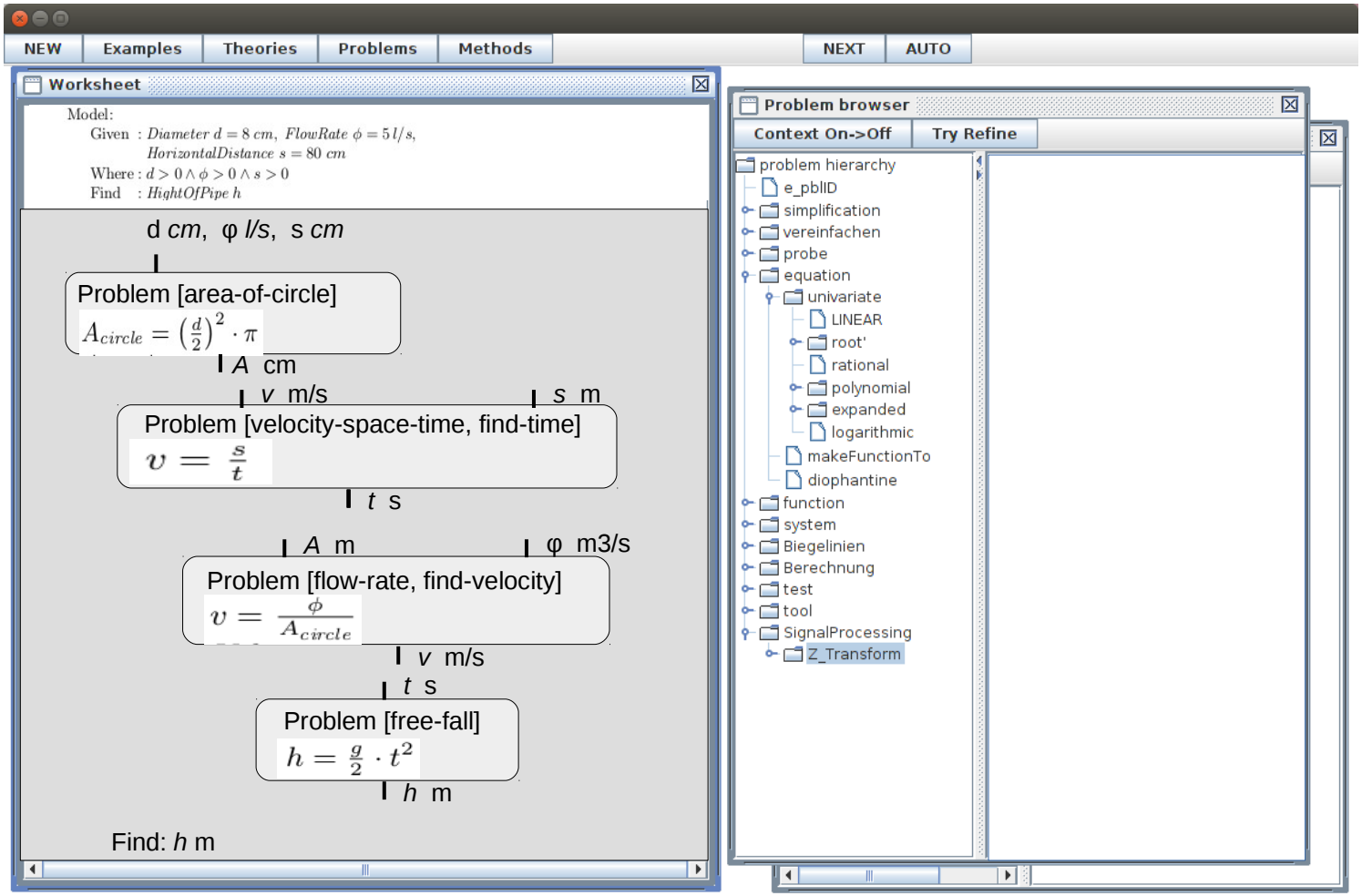
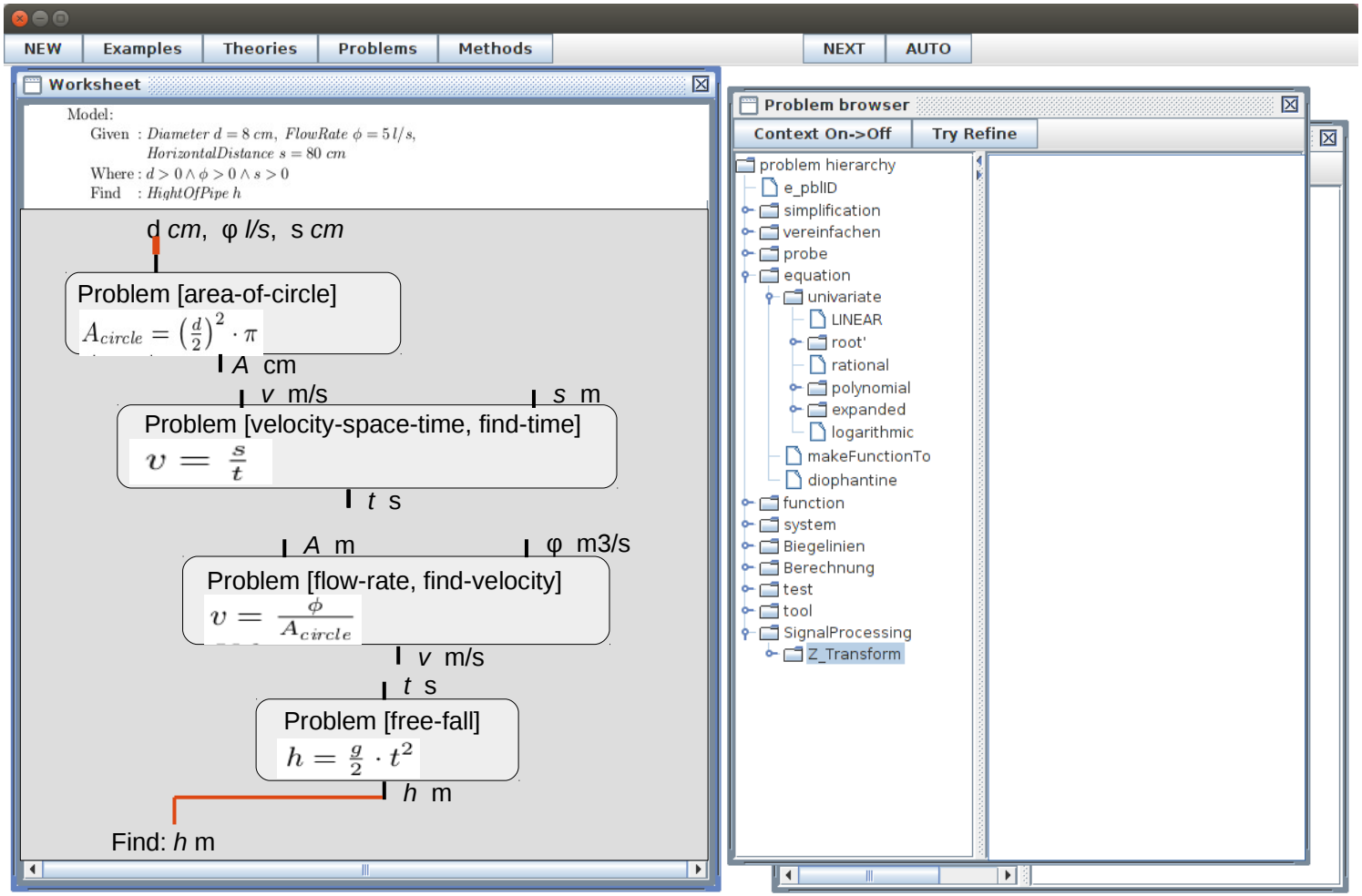


Connect “Given” and “Find”



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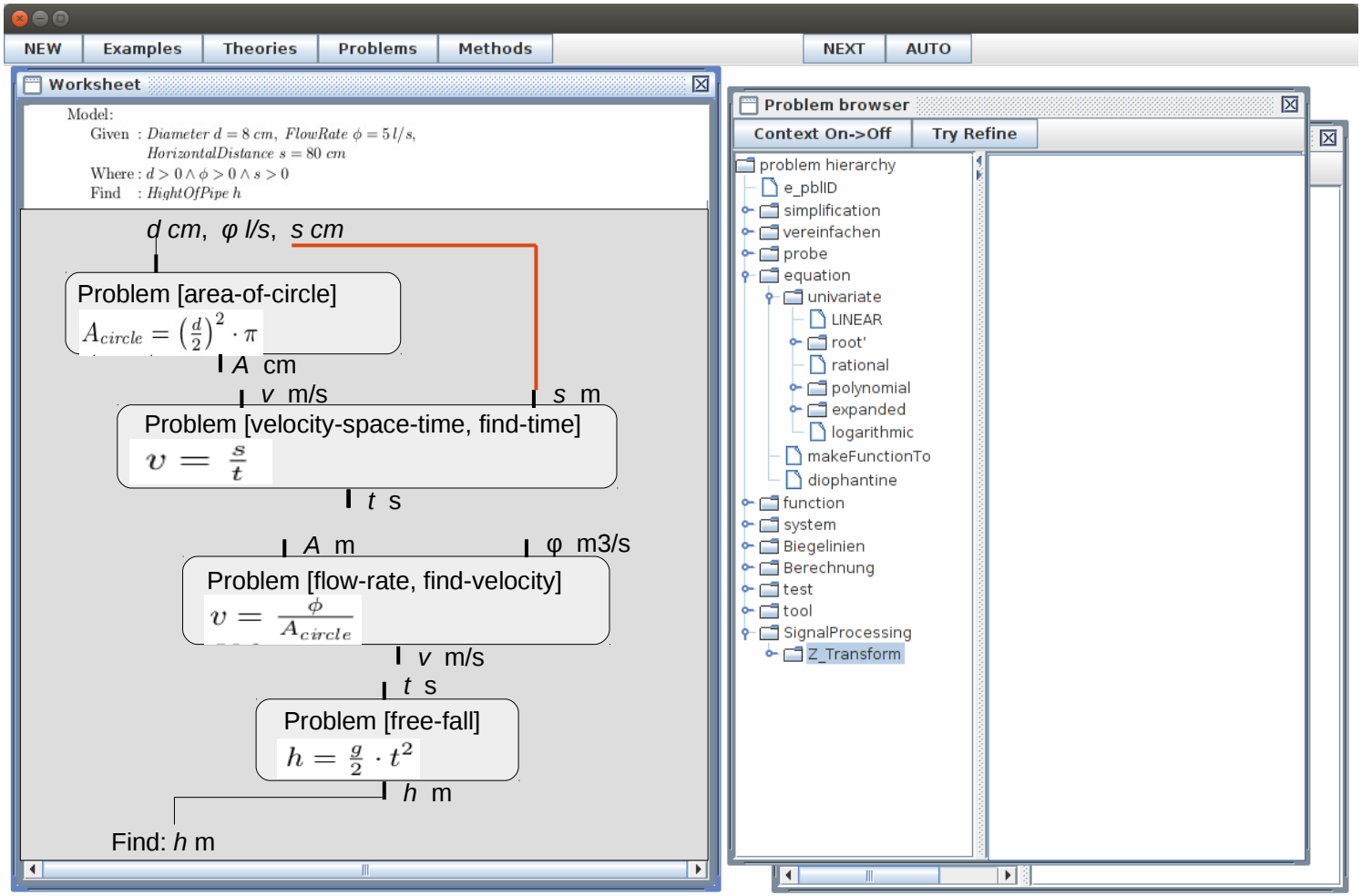

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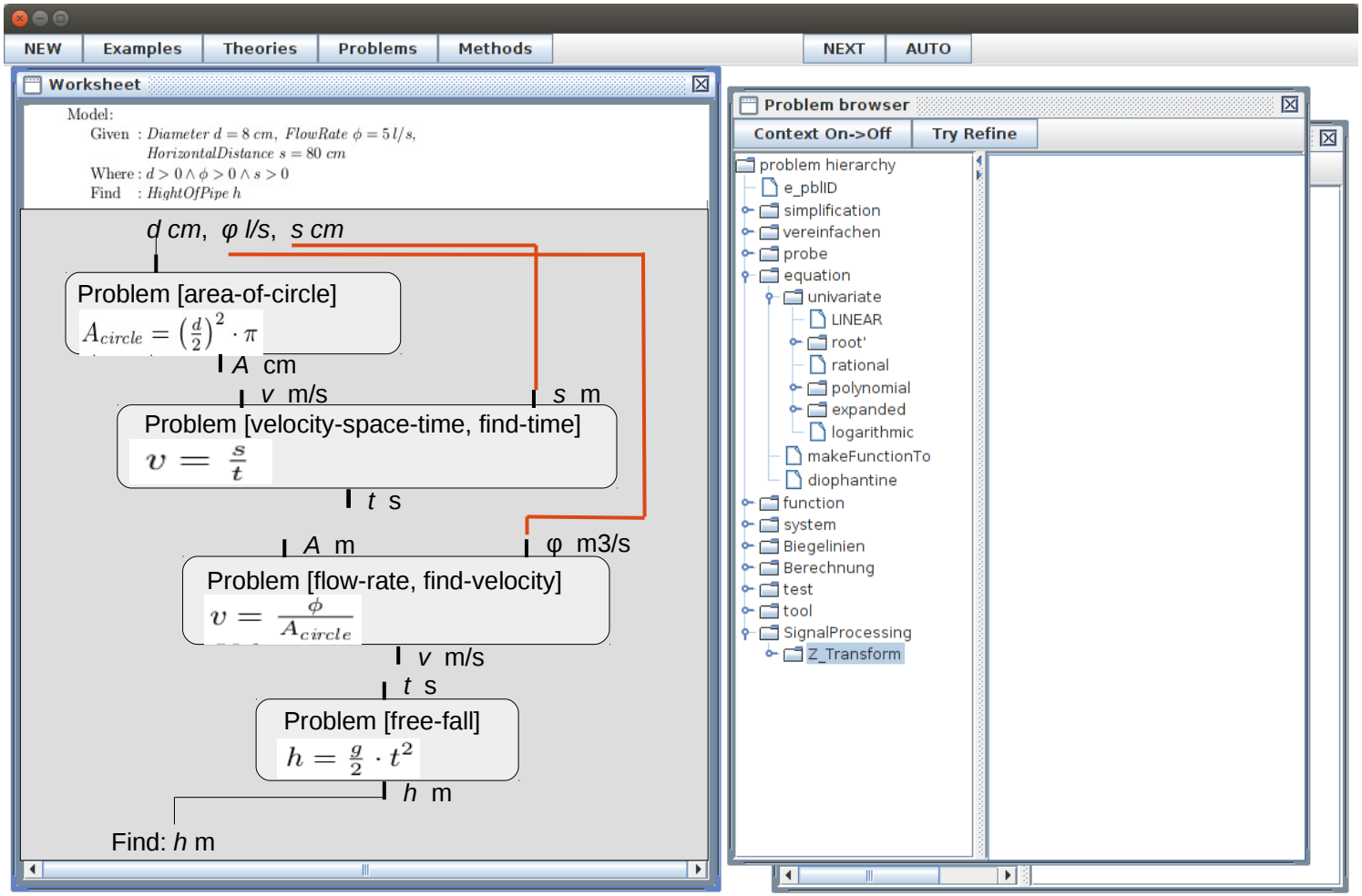

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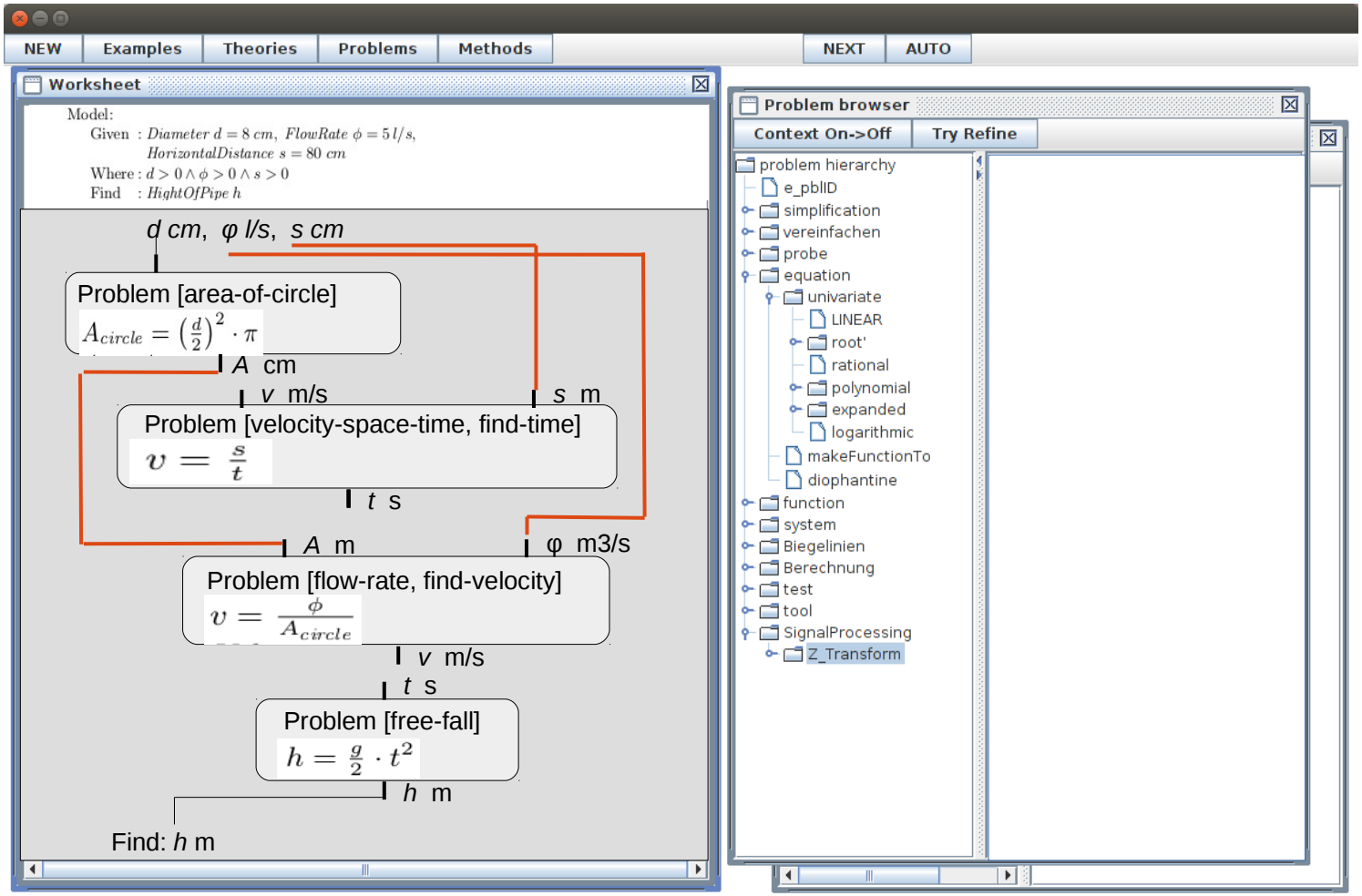

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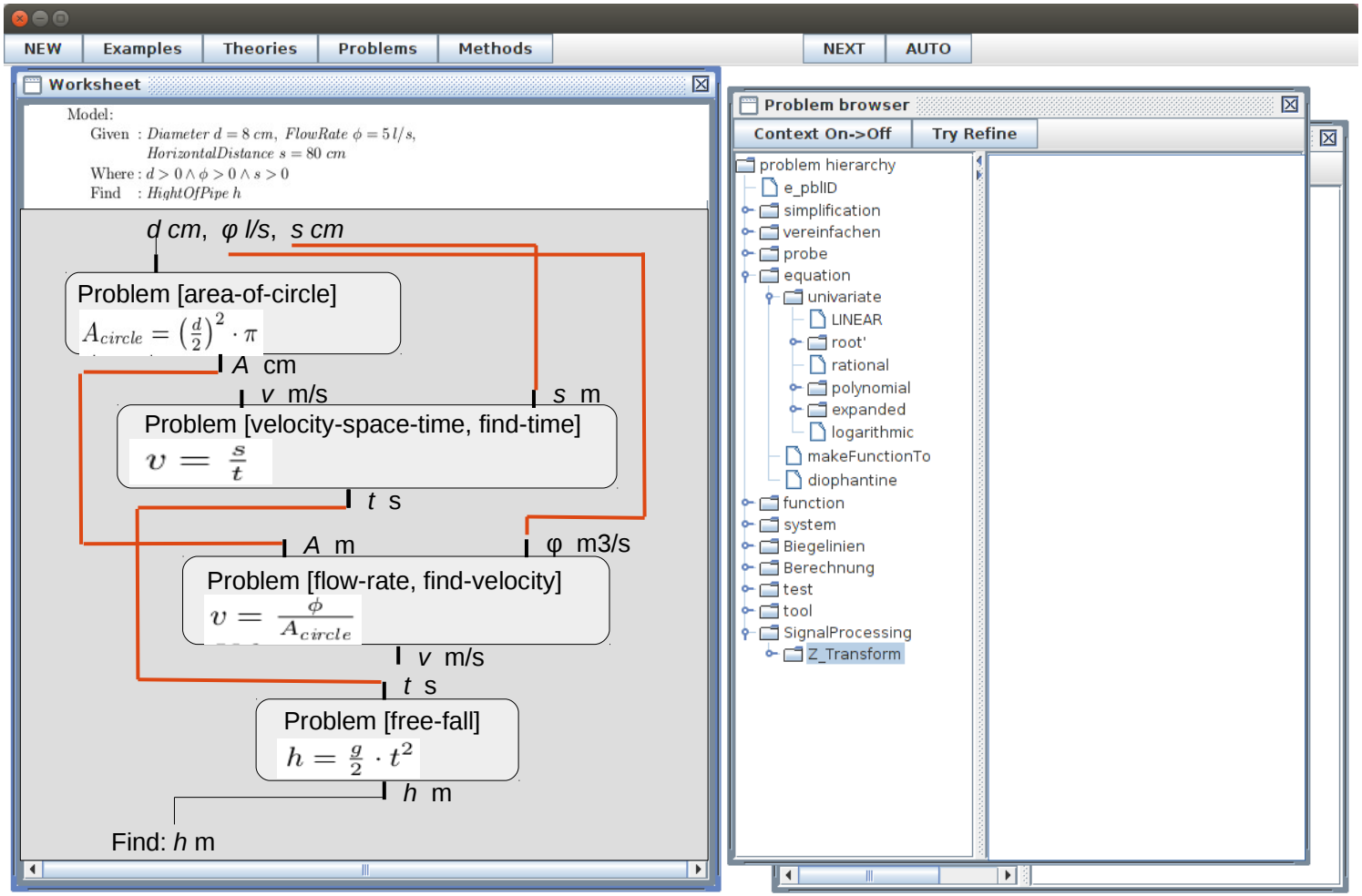
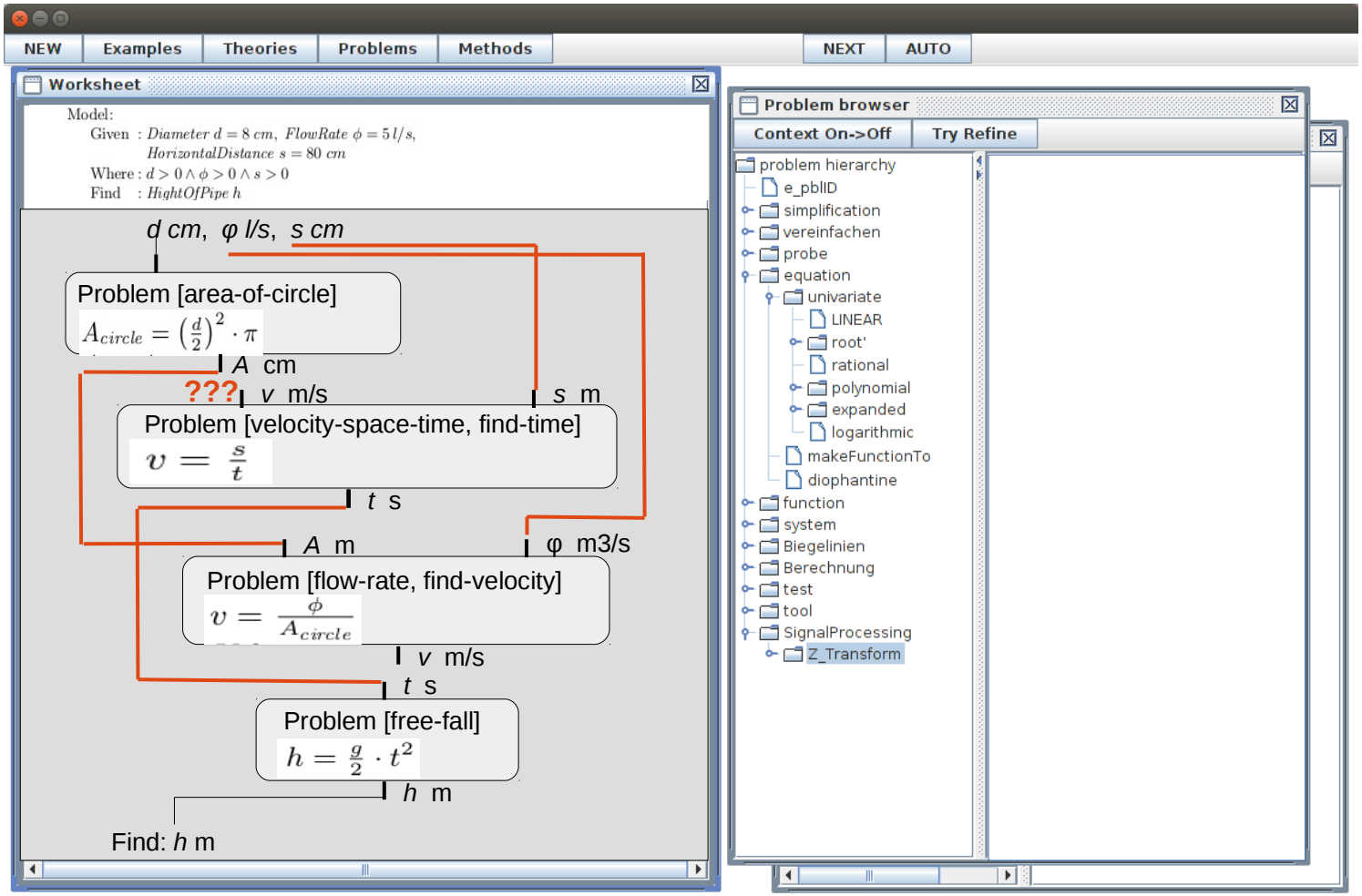
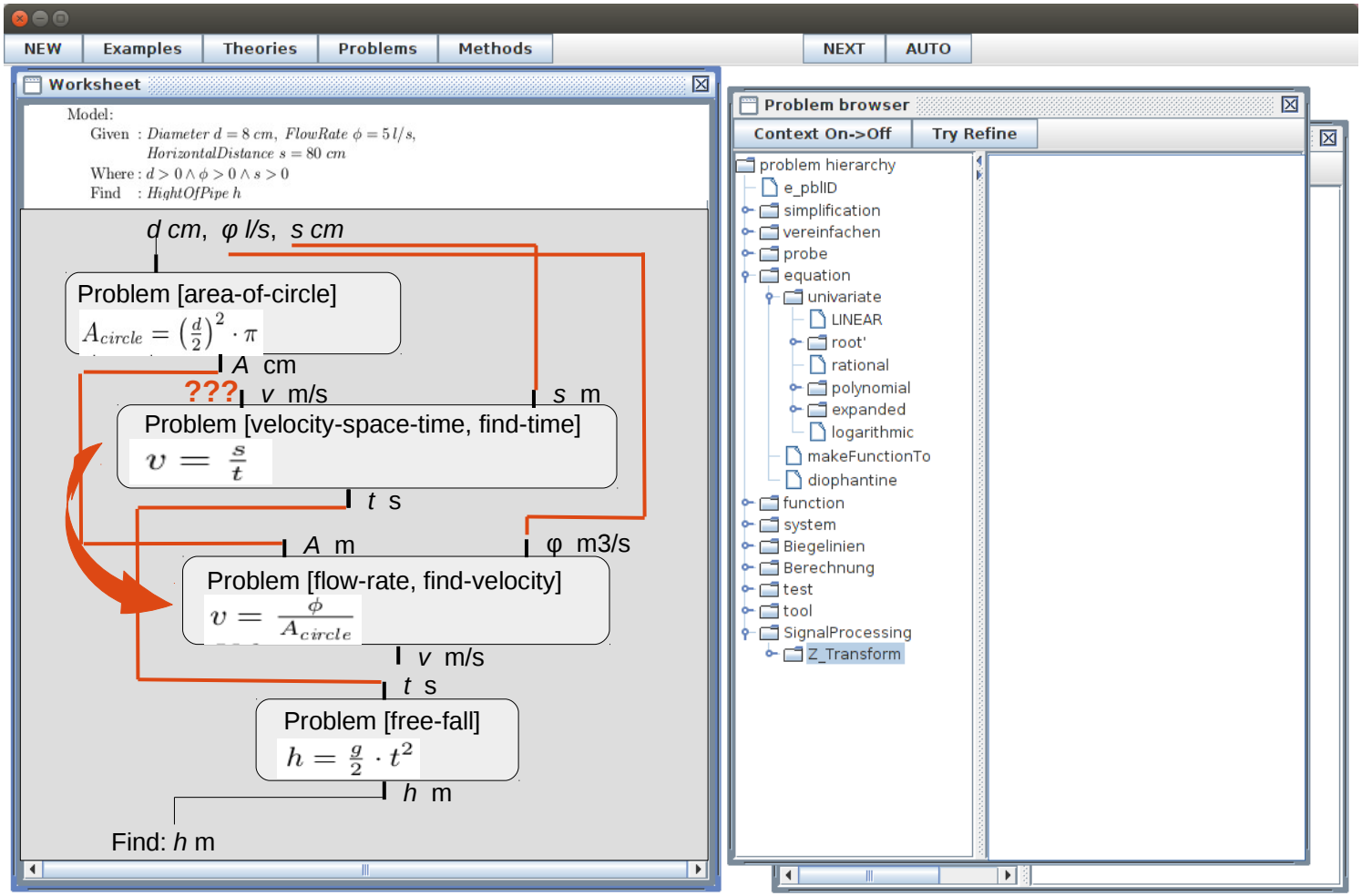


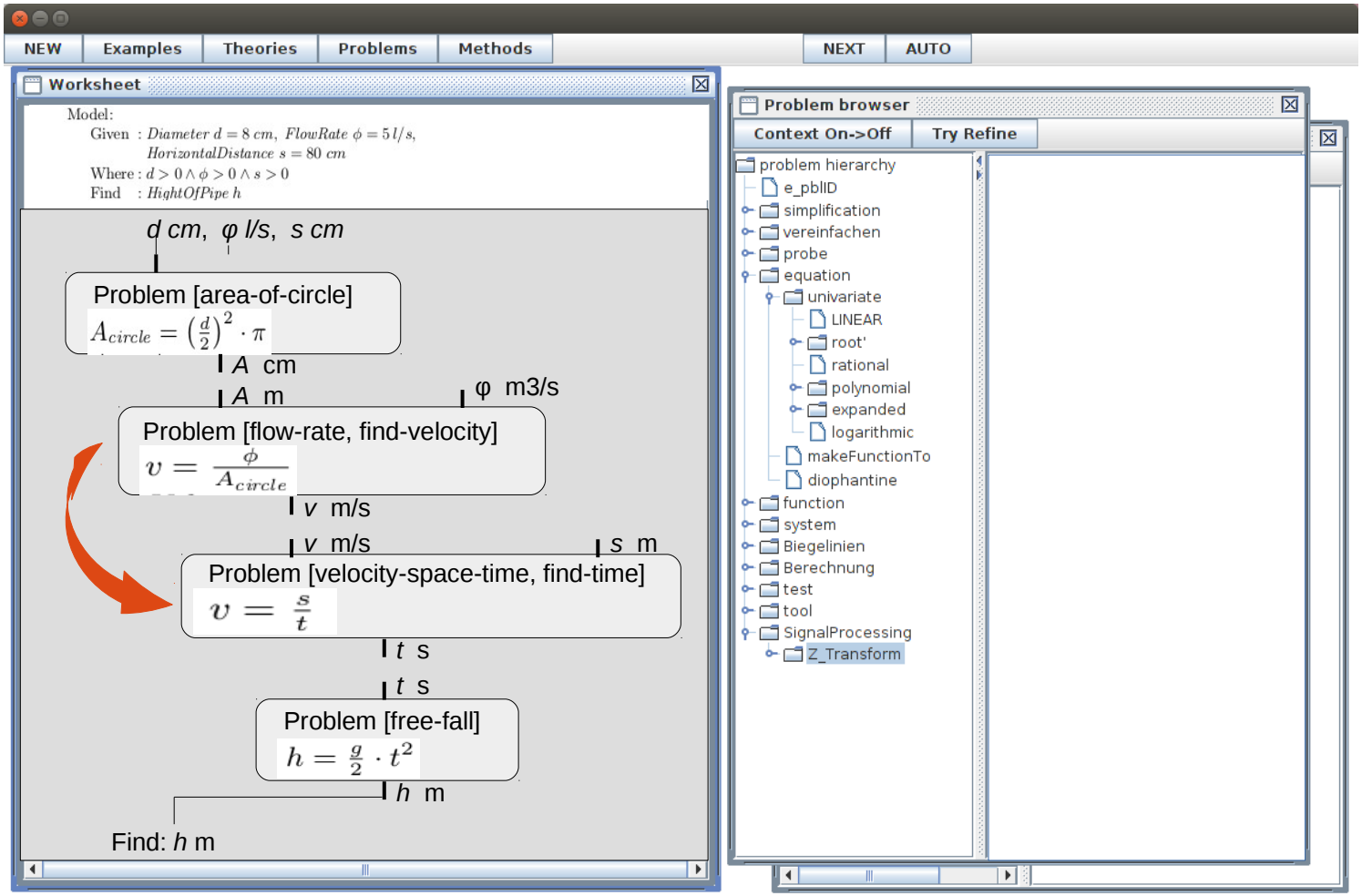
Dangling connection ???



Rearrange sub-problems



Flipped two sub-problems



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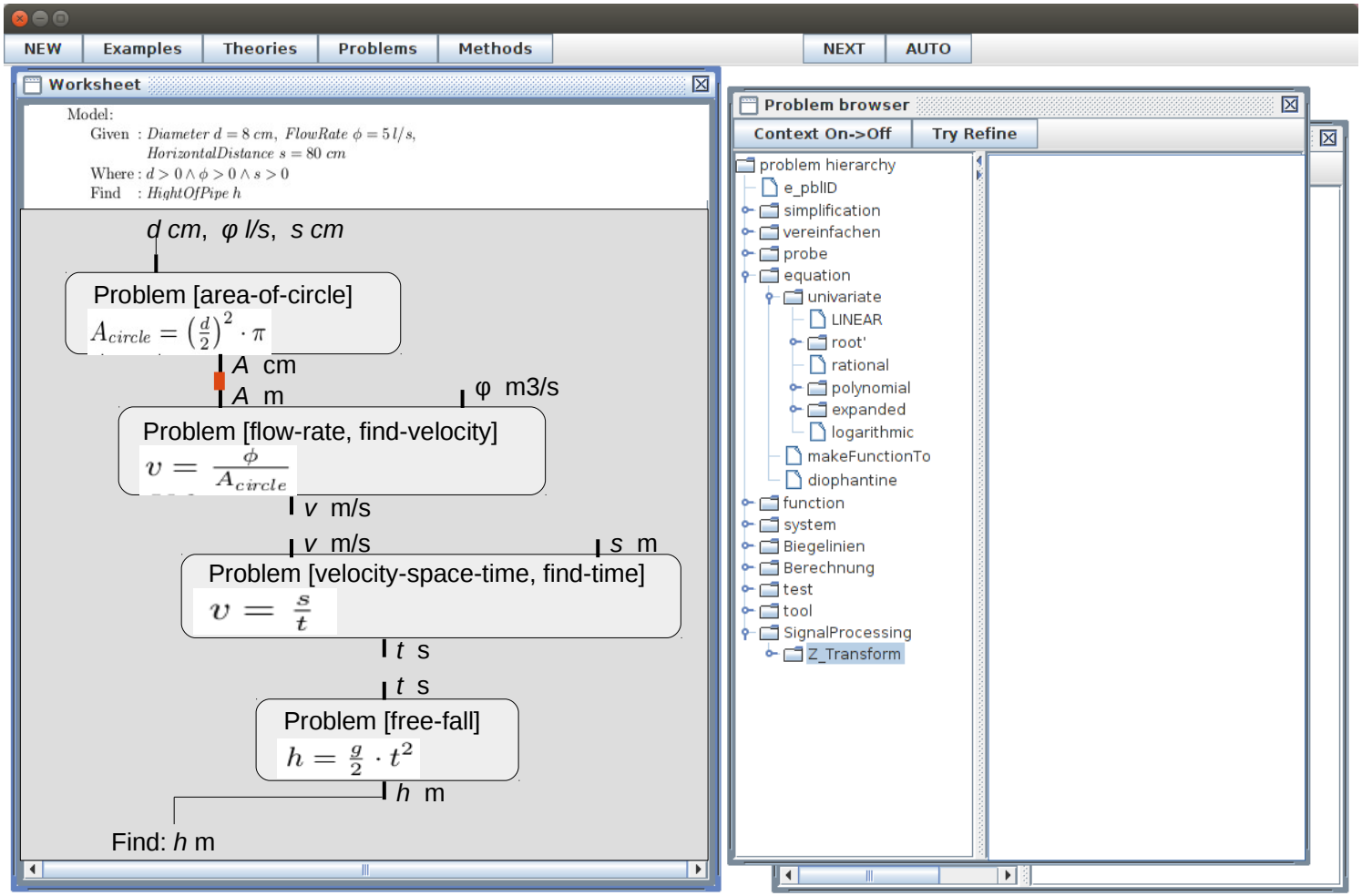

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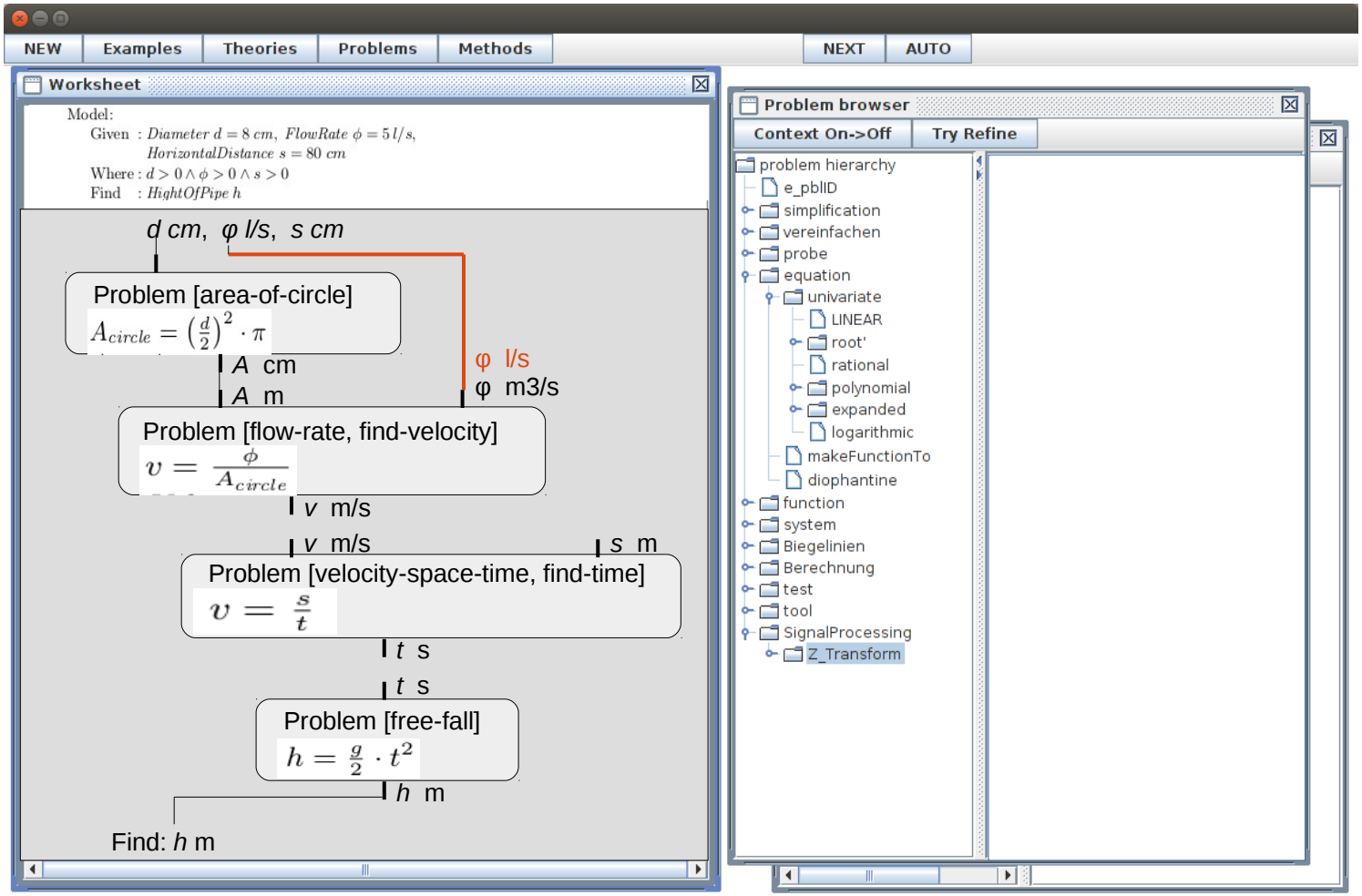

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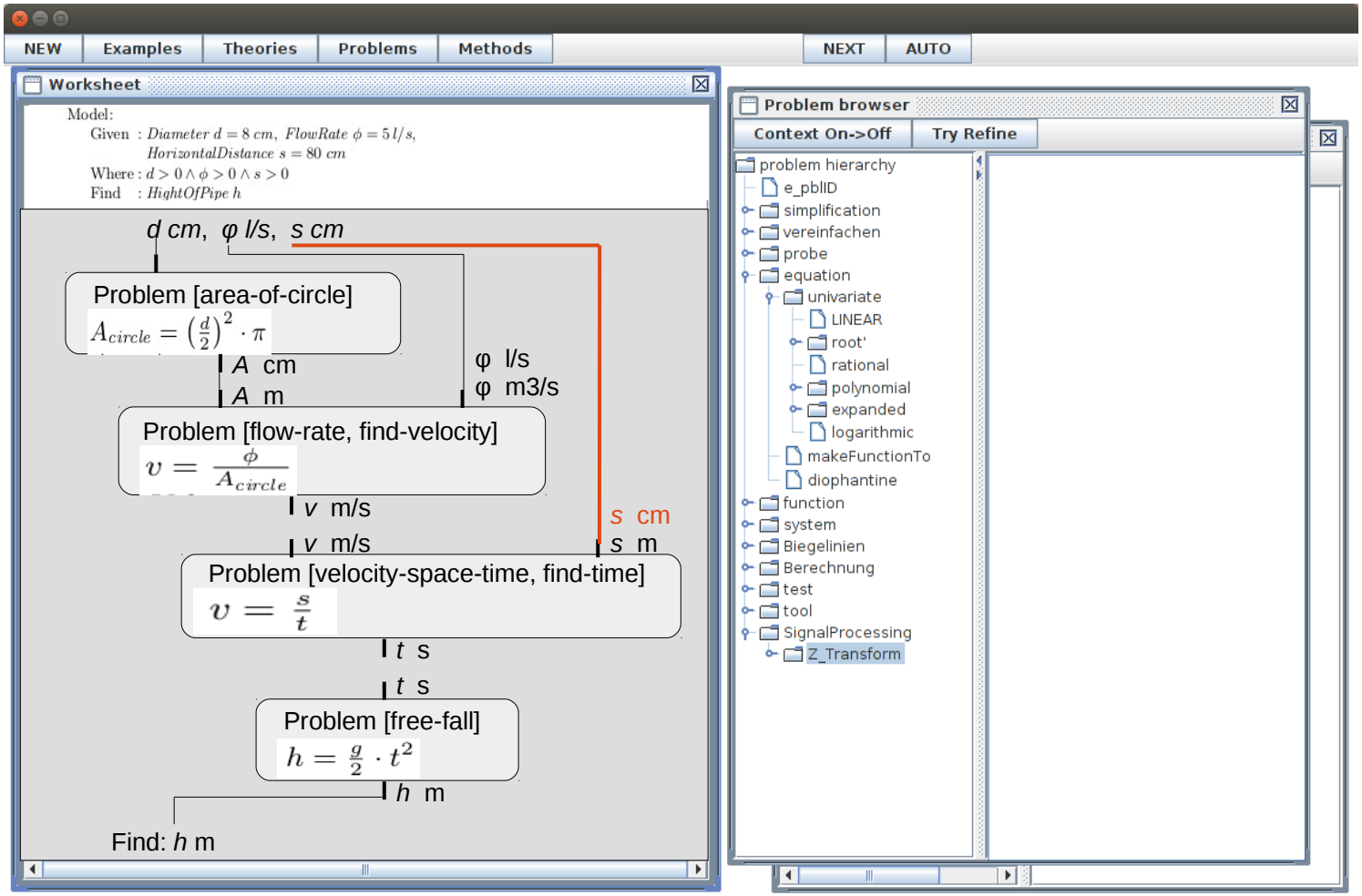

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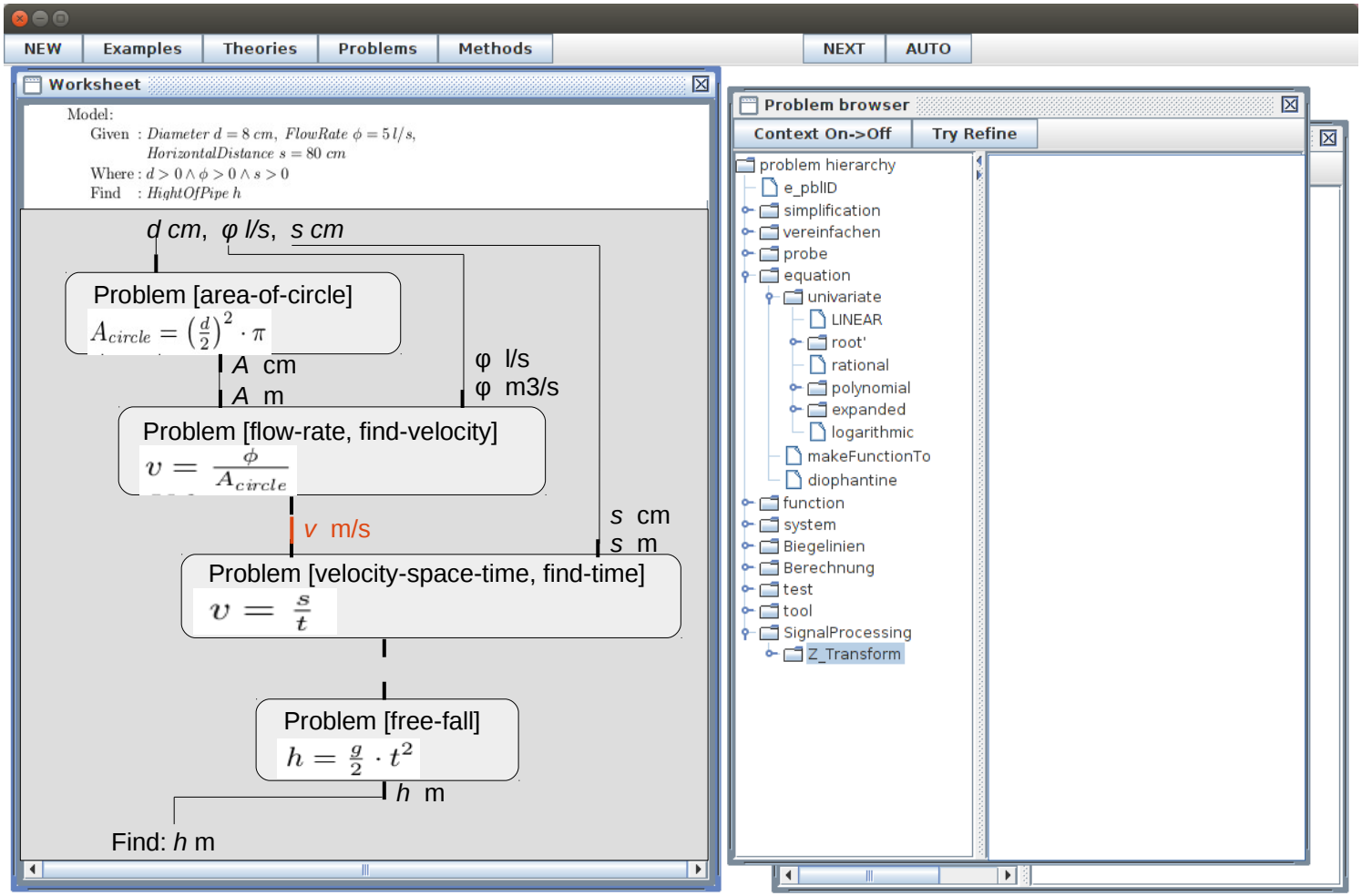

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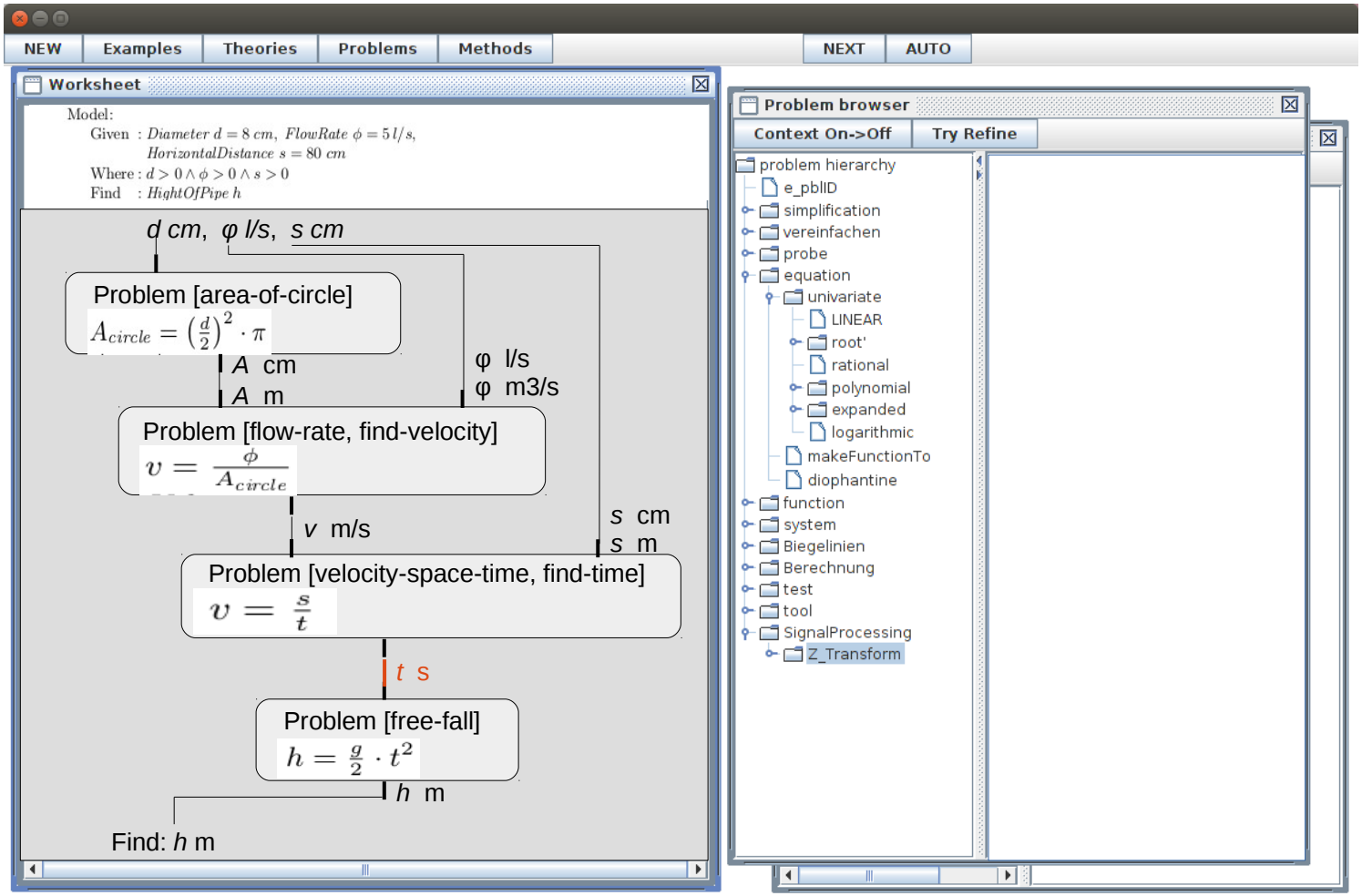
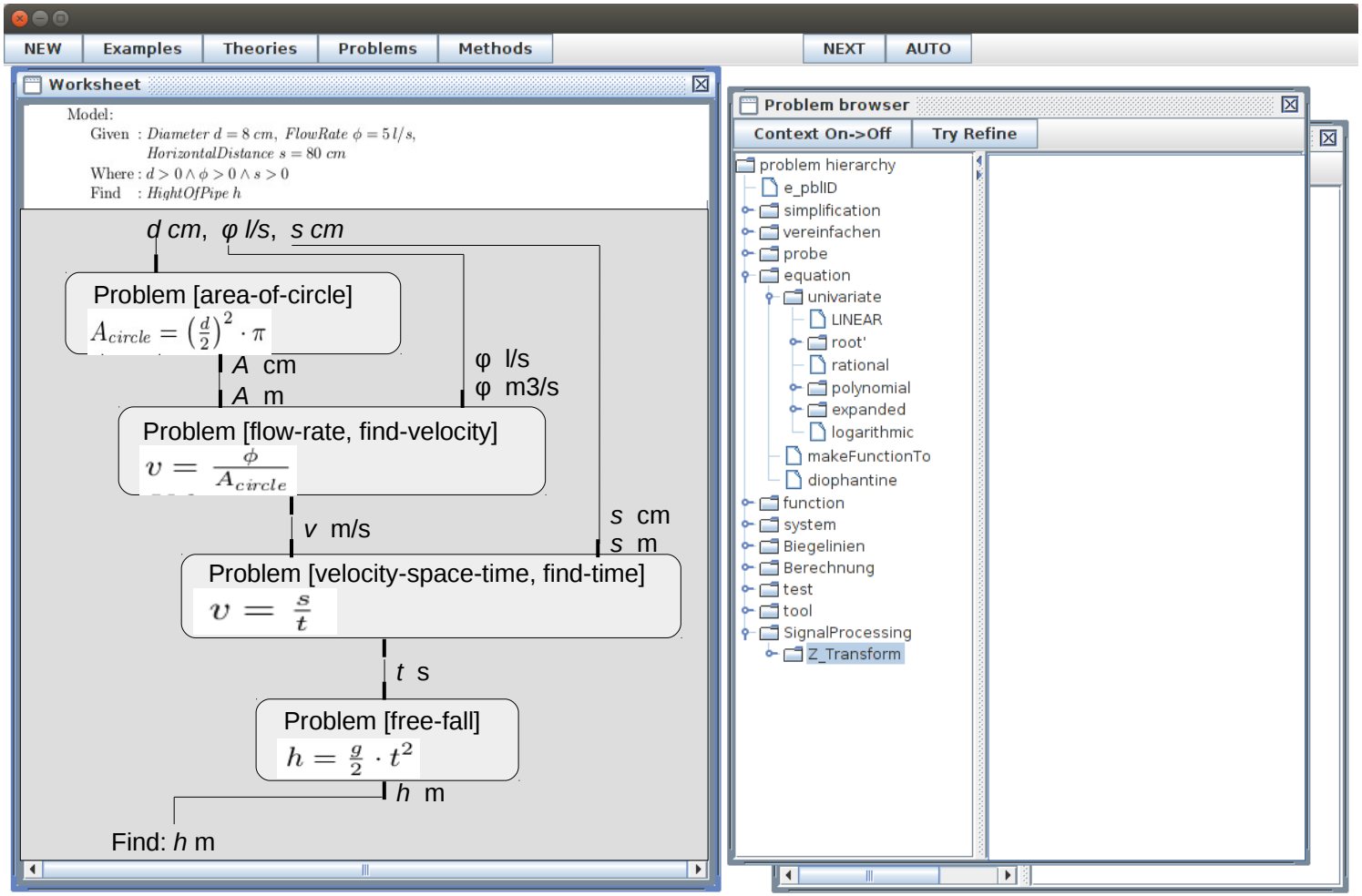
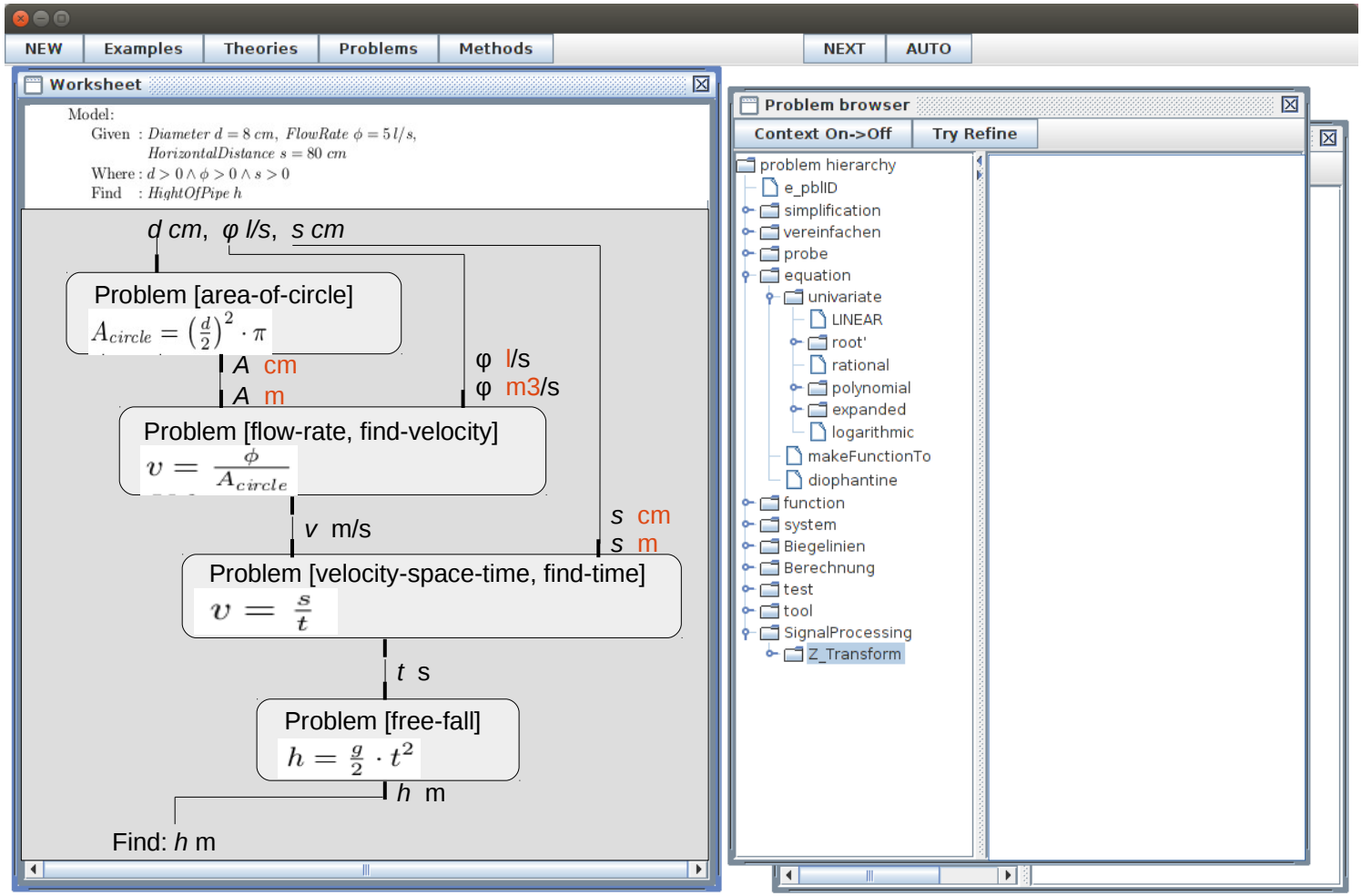


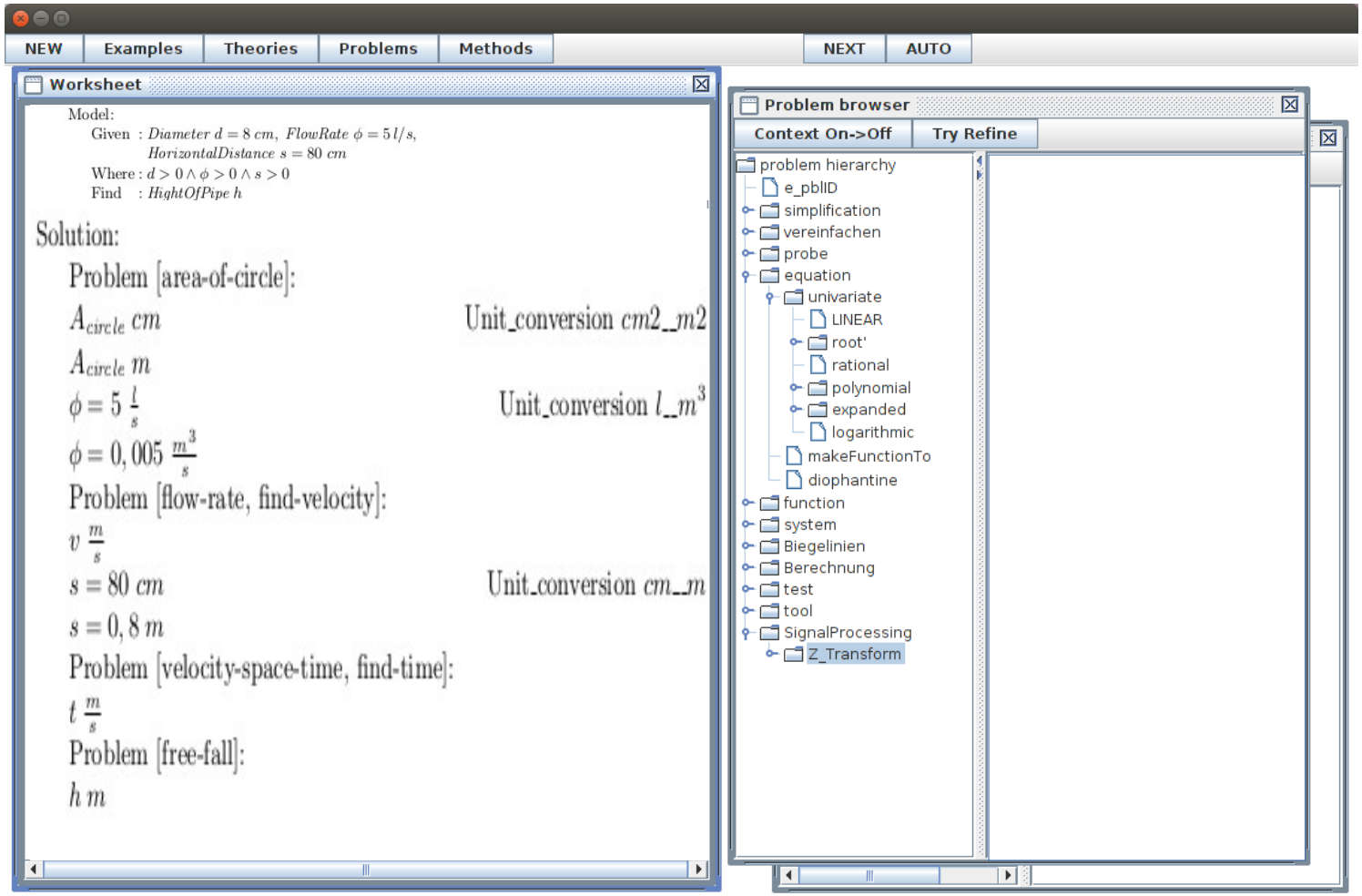
All connections finished



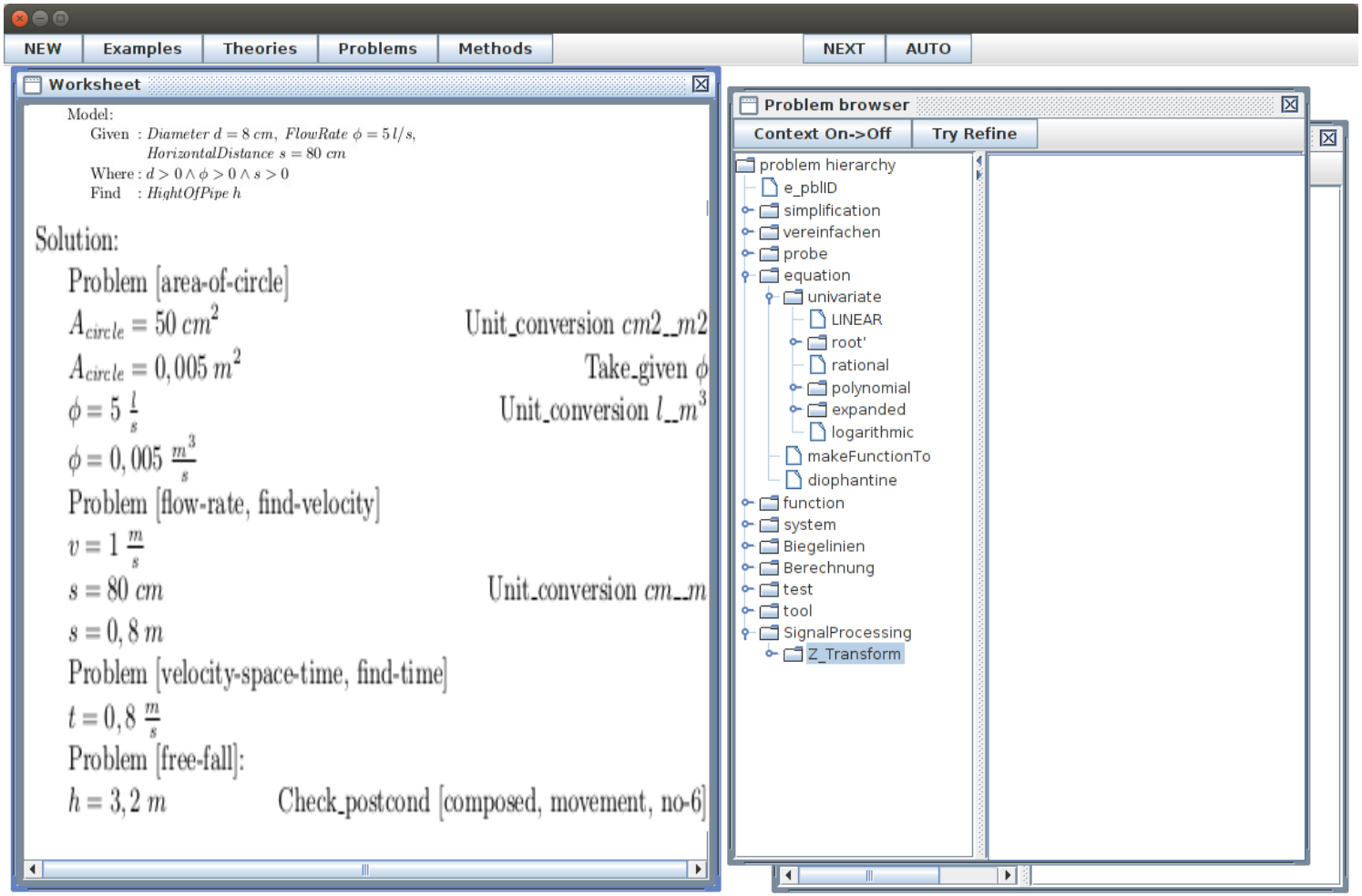
Care about unit conversions



Solution with units only



Solution complete



Outline

“Next step guidance” . . .

- . . . in specifying a problem:

  If **data for each variant** for constructing a specification
  (one variant shown above) are given,
  then the system can guide the student in completing
  a specification

- . . . in step-wise constructing a solution:

  If a **program** describes how to solve a problem
  defined by a formal specification,
  then this program run by **Lucas-Interpretation**

  - determines a next step (if requested by the student)
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Programming Language . . .

. . . for authors of mathematics knowledge.

Isabelle provides a **"function package"** for programming.
Added value of this implementation:

- syntax errors are indicated accurately
- type annotations shift into the initial signature
- less type annotations are required
- syntax highlighting specific for constants etc
- free variables on right-hand-sides are rejected

Students might watch progress within a solution
like in a debugger (on request).

Conclusions

TP technology provides mechanical explanations due to

- principal benefits
- added value of implementation
- and “next step guidance”

of various kinds on different language layers — all explanations come **on users’ request**!

Field tests will show, whether “systems that explain themselves” meet the promise to make learning mathematics a game like learning to play chess by software.

In order to get prototypes ready for field tests, understanding by stakeholders in STEM education is needed, private demos of Isabelle/*ISAC* are welcome.

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Thank you for Attention!