Designing tasks supported by GeoGebra Automated Reasoning Tools for the development of mathematical skills

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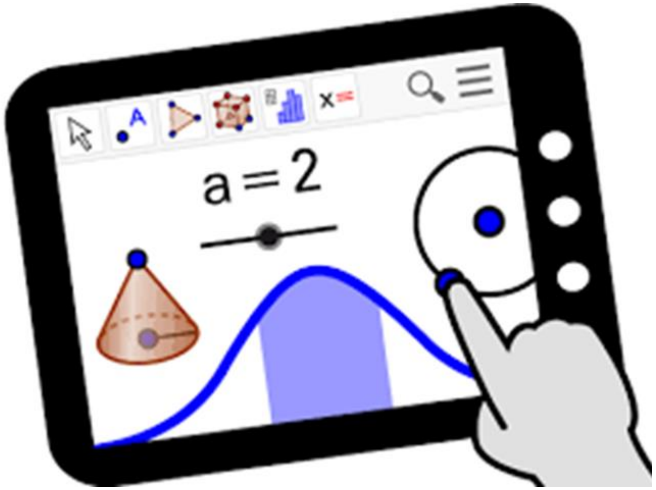
Our aims...

From

Automated reasoning tools for math learning
Use artifacts to achieve some didactic goals

to

Instrumented learning for math reasoning
Reconsider didactic goals of math education according to new computer tools



How?

Combining standard GeoGebra capabilities with the recently implemented (2016), tools and commands for the automatic reasoning (in an exact manner with mathematical rigour) over the free dynamic geometry software GeoGebra:

GGB-ART

- This novel technology, already imagined in the 80's (Howson and Wilson, 1986), attempts to address old, but still active, challenges in mathematics education (Sinclair, Bartolini, de Villiers and Owens, 2016).
- However, these new tools are still in an experimental phase regarding its use in the classroom, see Kovács, Recio, Richard and Vélez (2017) and Kovács (2018), for some pioneer references.

Reasoning assisted by GeoGebra

- Draw a construction and experiment by dragging the free objects → to conjecture and to convince oneself that certain properties are being verified
- Automated reasoning tools (GGB-ART) → to add a "geometric calculator" capable of finding conclusions with rigour

Visual abilities of GeoGebra + Autometed reasoning abilities through symbolic computtaion

Human reasoning

Deduction	Induction	Abduction
Rule: The diagonals of a parallelogram are perpendicular. *Case*: This polygon is a parallelogram. ➢ *Result:* The diagonals of this polygon are perpendicular.	*Case:* This polygon is a parallelogram. *Result:* The diagonals of this polygon are perpendicular. ➢ *Rule:* The diagonals of a parallelogram are perpendicular.	*Rule:* The diagonals of a parallelogram are perpendicular. *Result*: The diagonals of this polygon are perpendicular. ➢ *Case*: This polygon is a parallelogram.

And… why not take advantage of the three types of inference

Instrumental reasoning* by means of GeoGebra

Different milieu for reasoning with the machine:

- Visual: draw and move in the geometric view of GGB.
- Numerical: check for dimensions, coordinates, equations, etc. in the algebraic view of GGB.
- Algebraic: operate with polynomial expressions (implicit or parametric) using the CAS of GGB.
- Mathematical: rigorous deductive reasoning using GGB-ART.

*We refer to P. Richard's talk next Friday

GeoGebra automated reasoning tools (GGB-ART)

What is GG-ART?

GeoGebra Automated Reasoning Tools (GGB-ART) are a collection of GeoGebra 5.0 tools and commands ready to automatically derive, discover and/or prove geometric statements in a dynamic geometric construction*

* A tutorial of the GGB-ART is available at https://github.com/kovzol/gg-art-doc

1. Draw and conjecture using classical GeoGebra features

- Start by drawing a geometric construction in GeoGebra.
- GeoGebra has many ways to promote investigating geometrical proprieties:
  - Visually we can investigate by dragging the free objects by setting the trace of a constructed object or by using sliders.
  - Use Relation tool to obtain common properties to some objects.
  - Use Locus tool to draw the trace of an object for while another one moves on a path.

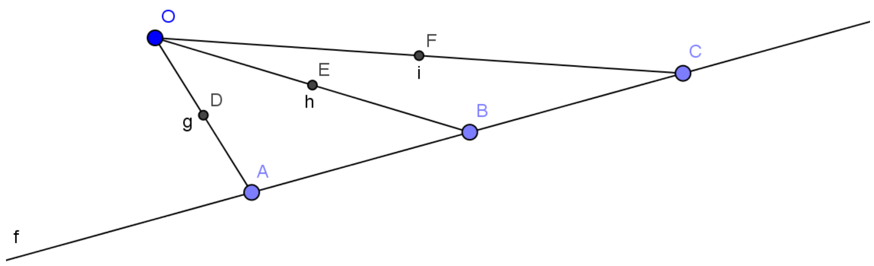
2. Deriving, discovery and proving using new commands with symbolic support

- **Relation** tool and command give information about the geometric relations between two or more objects.
- **LocusEquation** command calculates the equation of a locus and plots this as an implicit curve. It is used to discover complementary, necessarily, hypotheses for the conjectured geometric statement to become true
- **Prove** and **ProveDetails** commands decide if a statement is true in general and, eventually, give some additional conditions for its truth (low level GG tools).

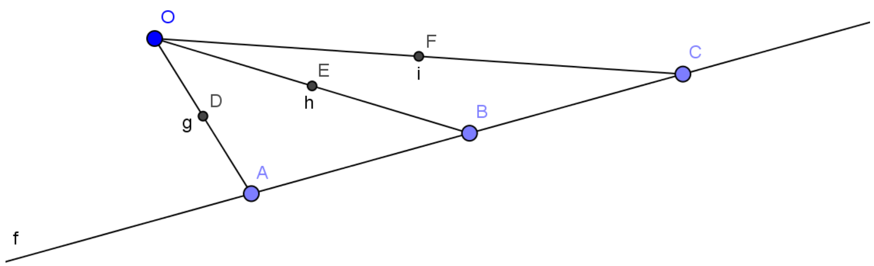
2.1. The Relation tool and command

Relation[<Object>, <Object>]
Relation[<Object>, <Object>, <Object>]
Relation[<Object>, <Object>, <Object> , <Object>]

Example: Take three aligned points A, B and C and a free point O, and the midpoints D, E and F of OA, OB and OC. Ask for relations between points or between segments or between lines in the configuration



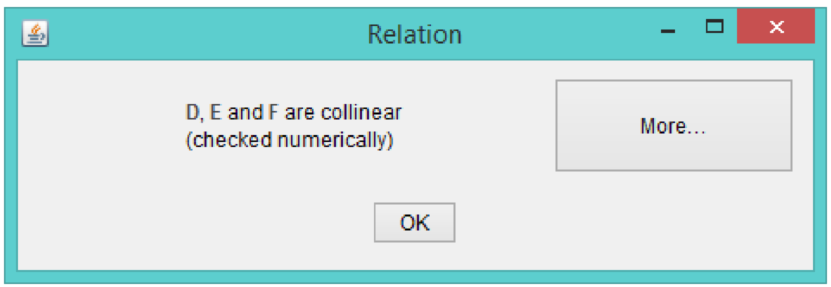
2.1. The Relation tool and command

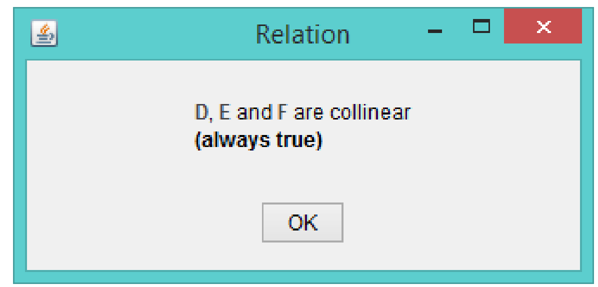


Are D,E,F aligned?

We can draw line DE and visually check that F is in this line, but we can also ask:

Relation[{D,E,F}]





Now it seems that lines through ABC and DEF are parallel: Ask GeoGebra again!!

2.1. The Relation tool and command

What is possible to check with Relation?

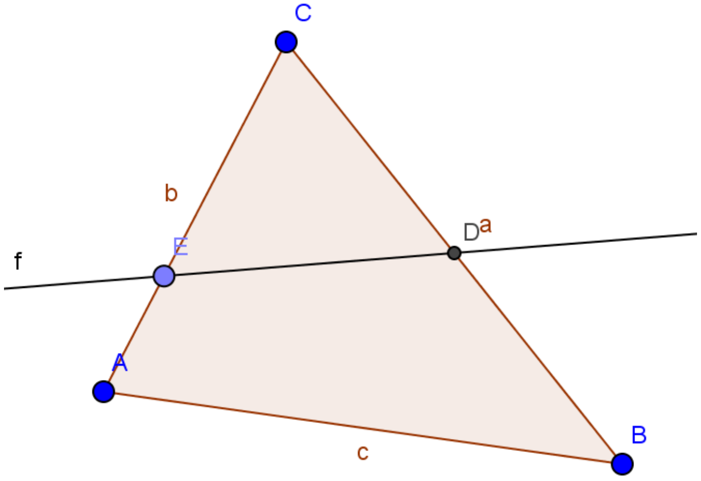
- The Relation command allows the user to find out numerically (that is, for the drawing with assigned coordinates) whether
  - two lines are perpendicular,
  - two lines are parallel,
  - two (or more) objects (points, lengths, areas) are equal,
  - a point lies on a line or conic,
  - a line is tangent or a passing line to a conic,
  - three points are collinear,
  - three lines are concurrent (or parallel),
  - four points are concyclic (or collinear).
- Some of these checks can also be performed symbolically, that is, the statement can be verified rigorously for the general case (with arbitrary coordinates).

2.2. The LocusEquation command

Implicit locus of a free point, or a point moving in a path, into a construction such that some property holds.

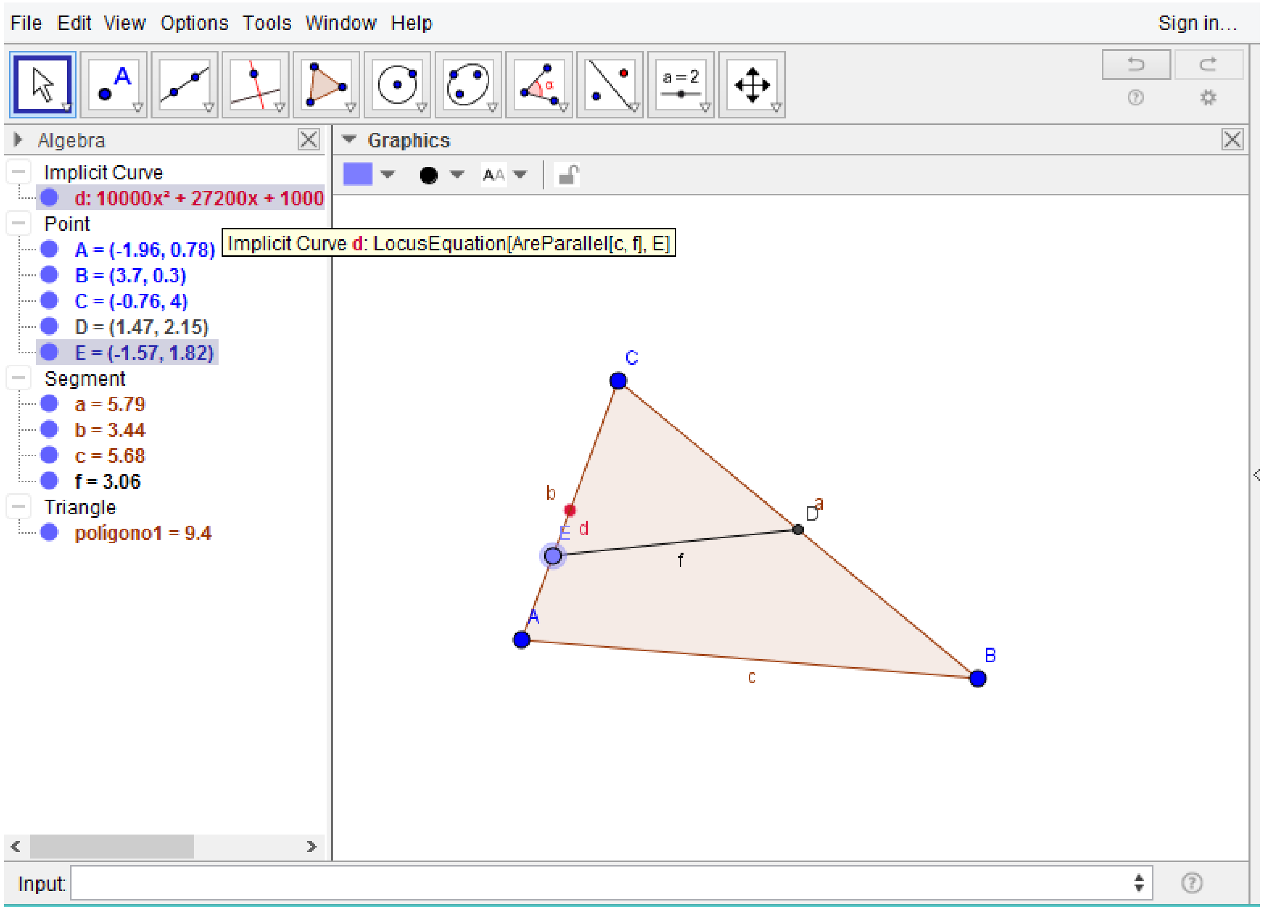
LocusEquation[<Boolean Expression>, <Point>].

Example 2: Consider a triangle with vertex *A, B, C* and sides *a, b, c*. Take *D* the middpoint of *a* and *E* any point on side *b*.



Question 1: Where must we place E such that segments *AB* are *DE* are parallel?

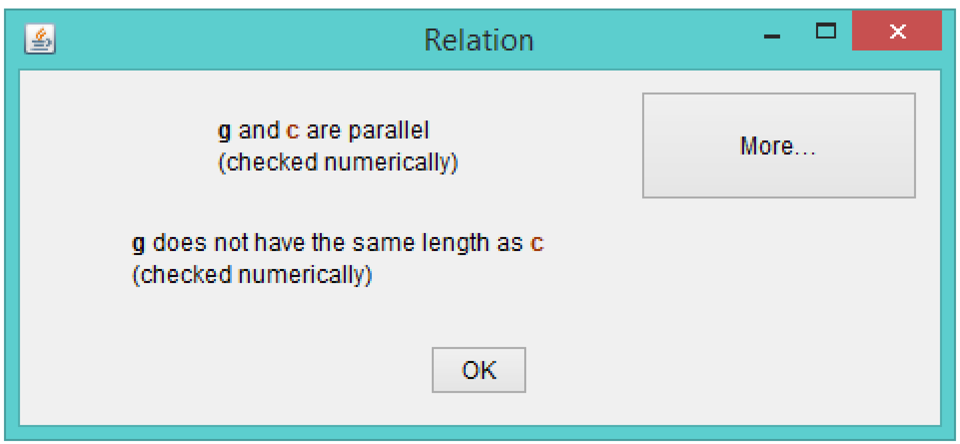
LocusEquation[AreParallel[f, c], E]

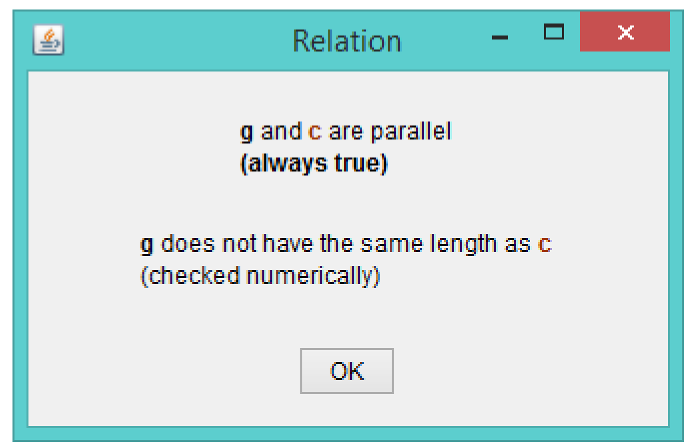


Drag the free objects and conjecture that object d must be the midpoint of b.

Question 2: . To confirm this conjecture create midpoint F of segment b and prove now that they are parallel.
For that, make the objects E, f and d invisible by hiding them.
Join D and F by segment g.
Use the Relation tool to compare c and g

Relation[c,g]





2.3. The Prove and ProveDetails command

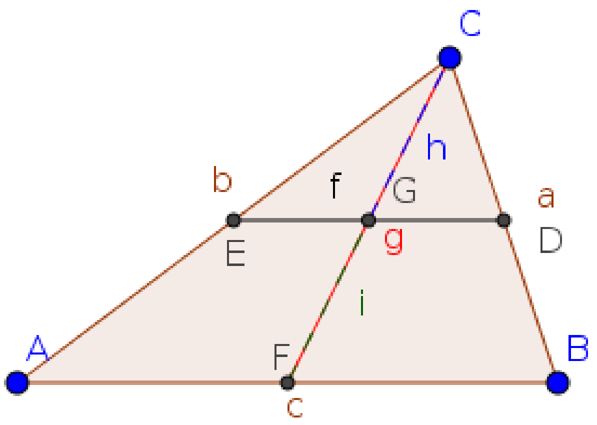
Prove[<Boolean Expression>]
ProveDetails[<Boolean Expression>]

The Prove command decides if a geometric statement is true. It has three possible outputs:

- *True:* the statement is always true, or true under some non-degeneracy conditions
- *False:* means that the statement is false in general
- *Undefined*: means that GeoGebra cannot decide because of some reason.

The ProveDetails command has similar behavior like the Prove but it may provide information about non-degeneracy conditions

Question 3: Use Prove and ProveDetails for proving conjecture “h and i are equal”



Prove [h==i] returns *true*

ProveDetails[h==i] returns *{true,{“AreCollinear[A,B,C]”}}*

Designing tasks supported by GGB-ART

Experience with prospective mathematics teachers

1. Overview

	Québéc-Canadá	Cantabria-Spain	Madrid-Spain
University	Montréal	Cantabria	Nebrija
Professor	P. Richard	T. Recio	M.P. Vélez
Level	Undergraduate	Graduate	Graduate
Studies	Degree	Master	Master
Mode of teaching	Classroom	Classroom	Online (videoconferences)
Previous training	Secondary school	Degree in Math, Engineering or Economics	Degree in Math, Engineering or Economics
Number of students	20	15	40

2. Methodology

- Objective: to study the potential of GGB-ART for the development of emergent professional skills in prospective math teachers
- Task: to propose experiential activities based on GGb-ART to solve outside the classroom in groups of 2 or 3.
- Development:
  - Design tasks
  - Training session (2-3 hours) in using GGb-ART
  - Group working (2-3 weeks) on the proposed activities (2-3 students)
  - Analysis of results and conclusions

3. Activities

3.1. Exercise 1

Guided activity with short questions

- Objectives: To foster the instrumental genesis and to detect limits in the use of dynamic geometry
- Developed in three steps:

Experimentation—Genesis--Reflection

Ejercicio 1 *El arte de lo posible*

Consideremos el siguiente problema:

¿Qué condiciones debe verificar un triángulo para que dos bisectrices de sus ángulos interiores sean perpendiculares? Justificar.

Y responder a las cuestiones que siguen.

First results:

- In general, objectives have been achieved
- Certain limitations of visual reasoning (continuity, discretization, infinite precision,...) have been detected by students
- Development of professional skills for this exercise is positive: students are stimulated.

1. Método experimental.

a) Trazar con GeoGebra un triángulo ABC y construir las bisectrices f y g de los ángulos en los vértices B y C respectivamente. Desplazar el vértice A intentando que f sea perpendicular a g. ¿Qué posición de A parece que satisface la condición de perpendicularidad?

b) Etiquetar con O el punto de intersección de las bisectrices y sea α el ángulo BOC. Activar la cuadrícula y asegurarse de que la captura de punto está activada. Situar los vértices del triángulo en los puntos: $A(0, 3)$, $B(-1, -1)$ y $C(1, -1)$. Cambiando la escala de la cuadrícula en cada caso, mover el punto A y anotar los valores correspondientes para α en la siguiente tabla:

Experimentation

$A(x_A, y_A)$	α
(0, 3)	
(0, 10)	
(0, 100)	
(0, 1 000)	

¿Qué tendencia se observa en los valores de α? Ir a la ventana Vista algebraica y reemplazar y_A primero por 10^5 y luego por 10^6. ¿Qué sugiere el valor de α?

2. EL lugar de puntos.

a) Volver a la ventana Vista gráfica de GeoGebra y situar los vértices del triángulo como sigue: $A(0, 3)$, $B(-1, -1)$ y $C(1, -1)$. En la línea de comandos, escribir:

```
EcuaciónLugar(SonPerpendiculares(f, g), A).
```

Genesis

¿Qué significa la curva implícita `d: y=-1`? Justificar.

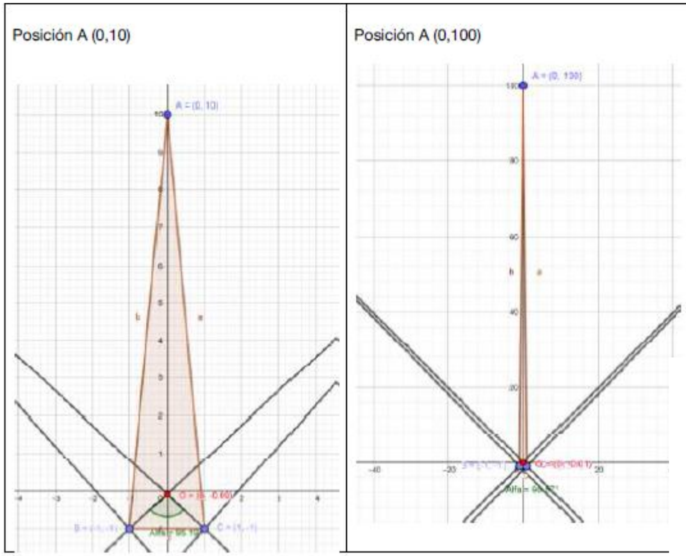
b) Aprovechando la captura de punto de la cuadrícula situar A sobre la recta d. ¿De qué tipo es el triángulo ABC resultante? ¿Cuánto mide el ángulo α? ¿Las rectas f y g son perpendiculares? Interpretar las respuestas teniendo en cuanta el problema inicial.

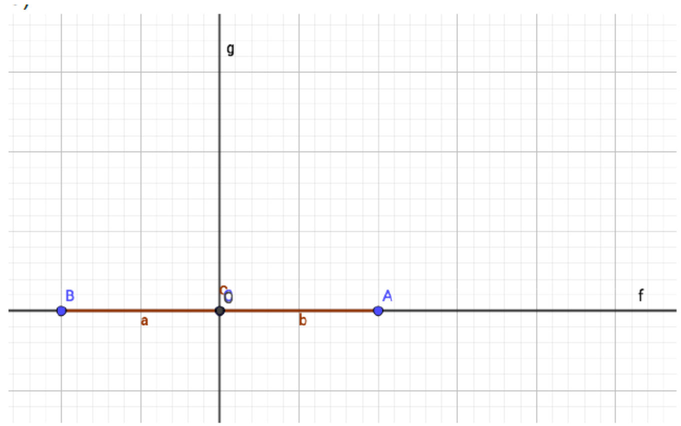
Reflection

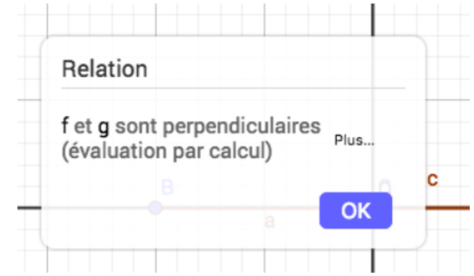
Inspirándose en la cita anterior, ¿podemos suponer que en un triángulo ABC muy, muy grande, enorme que sea isósceles en A, las bisectrices de los ángulos en los vértices de la base son perpendiculares? ¿Por qué?

e) ¿Qué opinas sobre la siguiente propiedad: «en un triángulo ABC, si el vértice A pertenece a una de las bisectrices en los vértices B o C, entonces dichas bisectrices son perpendiculares»? Justificar si esta propiedad es válida *visualmente* (por ejemplo situando A en la recta por BC o en el segmento BC y comparando después lo que se ve), *instrumentalmente* (invocando, si es posible, un comando como `Relación`, `Lugar`, `EcuaciónLugar`, `Envolvente`, `Demuestra` o `DemuestraDetalles`) y finalmente *matemáticamente*.

La mesure de l'angle α tend de plus en plus vers 90 °. Pour des y_a de 10^5 et 10^6 on obtient d'ailleurs une mesure de 90 °.



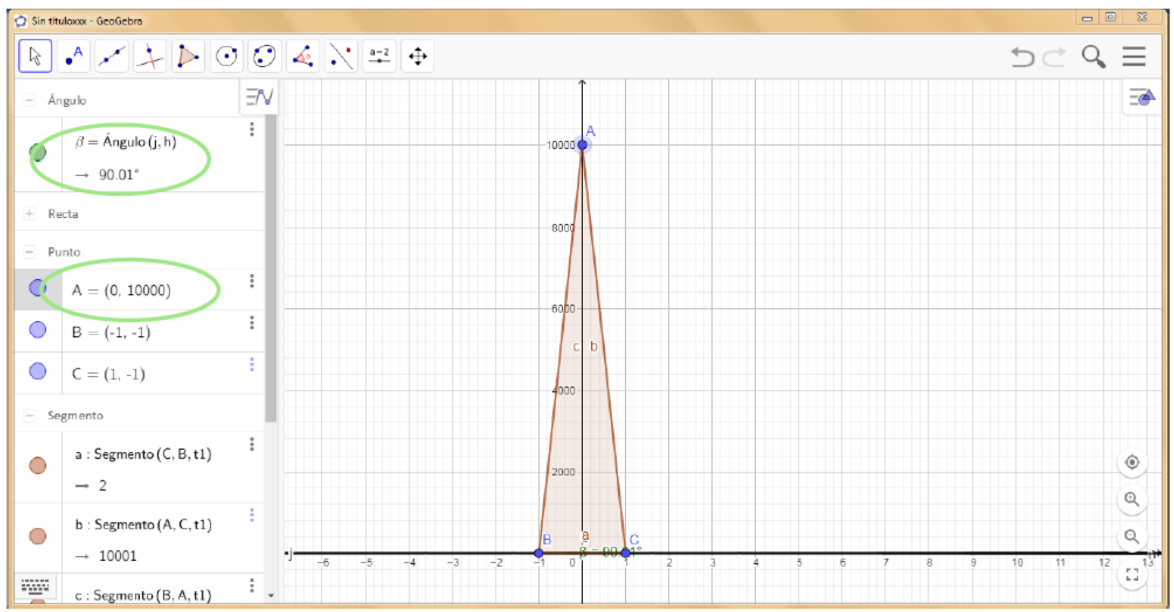




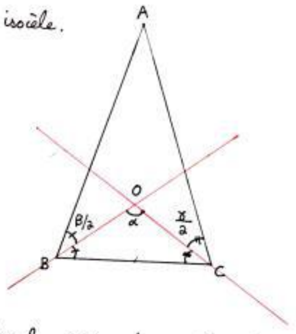
¿Qué tendencia se observa en los valores de α? Ir a la ventana Vista algebraica y reemplaz y_A primero por 10^5 y luego por 10^6. ¿Qué sugiere el valor de α?

El valor de alfa tiende a los 90°. En un principio baja con más rapidez conforme el vértice A s va alejando para luego ir teniendo variaciones más pequeñas.

Al reemplazar el valor de la altura o coordenada "y" por 10^5 y luego por 10^6 el valor e al adoptado por el programa es de 90° debido al redondeo que realiza.



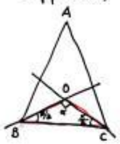
Soit un triangle isocèle. ABC



Soit β l'angle ABC et γ l'angle BCA.

- Puisque ΔABC est isocèle, on a que $\beta = \gamma$
- Construisons les bissectrices des angles β et γ : $\frac{\beta}{2} = \frac{\gamma}{2}$

- Supposons, par absurde, que α, l'angle entre les deux bissectrices, soit égal à $90°$



Considérons le triangle OBC, où O est le point d'intersection des deux bissectrices.

On a : $\alpha + \frac{\beta}{2} + \frac{\gamma}{2} = 180°$ (la somme des angles intérieurs d'un triangle)

$90 + \frac{\beta}{2} + \frac{\gamma}{2} = 180°$ (par hypothèse)

$\Rightarrow \frac{\beta}{2} + \frac{\gamma}{2} = 90° \Rightarrow \beta + \gamma = 180°$

Or, si $\beta + \gamma = 180°$, alors l'angle BAC serait nul, donc ABC ne serait plus un triangle. CONTRADICTION.
(cas dégénéré)

- Ainsi, dans un triangle isocèle, les bissectrices des angles à la base ne peuvent pas être perpendiculaires. CQFD

<u>Visuellement</u> : Si A est situé sur le segment BC, les bissectrices sont confondues avec le segment. Si A est situé sur la droite BC, mais non pas entre B et C, les bissectrices semblent perpendiculaires

<u>Instrumentalement</u> : Les bissectrices sont perpendiculaires si A est sur la droite d'équationLieu $y = -1$

<u>Mathématiquement</u> : Lorsqu'on déplace A vers C en le positionnant à l'extérieur du segment BC, l'angle ACB devient un angle de 180º, donc sa bissectrice est à 90º. Alors que l'angle CBA devient un angle de 0º bu que le triangle est aplati sur la base BC, donc

3.2. Exercises 3 and 4

Open activities

- Objectives: To promote the development of structured creativity using different means in the mathematical work to solve geometry problems.
- Students are asked to use at least two means (visual, numerical, instrumental, symbolic, etc.)

Exercice 3 *Résolu d'employer à dessein*

Répondre de deux façons différentes à chacune des questions suivantes en variant sur le moyen mis en oeuvre (visuellement, instrumentalement, mathématiquement, numériquement, symboliquement, etc.). Dans chaque cas, expliquer qu'elle est le niveau de conviction attaché à la solution (valeur épistémique faible, moyenne ou élevée).

Chosing your favourite reasoning milieu

2. Si Ω est le centre du cercle circonscrit d'un triangle, O le centre du cercle qui y est inscrit, H l'orthocentre et G l'isobarycentre (centre de gravité), quand les points Ω, O, G et H sont-ils alignés?

3. Dans le triangle ABC, on considère H, I et J les pieds des hauteurs, c'est-à-dire que HIJ est le triangle orthique de ABC. Quand le triangle orthique est-il équilatéral?

4. Construire deux cercles tangents et une droite tangente en A et en B à ces deux cercles. Existe-t-il une relation numérique entre le rayon de chaque cercle et la distance AB? Le démontrer.

5. Quels sont les rectangles dont la mesure de l'aire est égale à celle du périmètre, sans les unités bien entendu?

Generating activities

Exercice 4 *Mystères de la création*

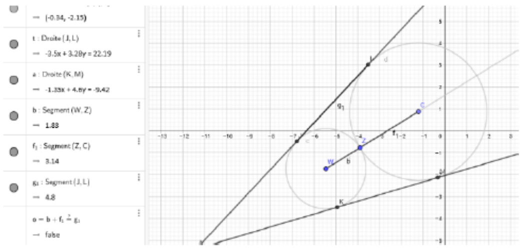
Proposer deux activités de découverte et trois activités de validation à l'interface de GéoGébra. Dans chaque cas, il faut prendre soin de bien indiquer le contexte du problème, les consignes particulières, le milieu didactique (logiciel, questionnaire) précis qu'interrogerait l'élève et avec qui il serait en interaction. On peut aussi fournir la carte d'identité de chaque activité.

First results:

- Students prefer to solve problems visually with dynamic geometry or deductively reasoning.
- We find a first difference between regions: the instrumental genesis is more activate for Spanish and the semiotic genesis for Québécois

4. Construire deux cercles tangents et une droite tangente en A et en B à ces deux cercles. Exist t-il une relation numérique entre le rayon de chaque cercle et la distance AB? Le démontrer.

Première méthode : preuve instrumentale (GéoGébra)



***valeur épistémique : faible (instrumental/visuel)**

Deuxième méthode : preuve mathématique

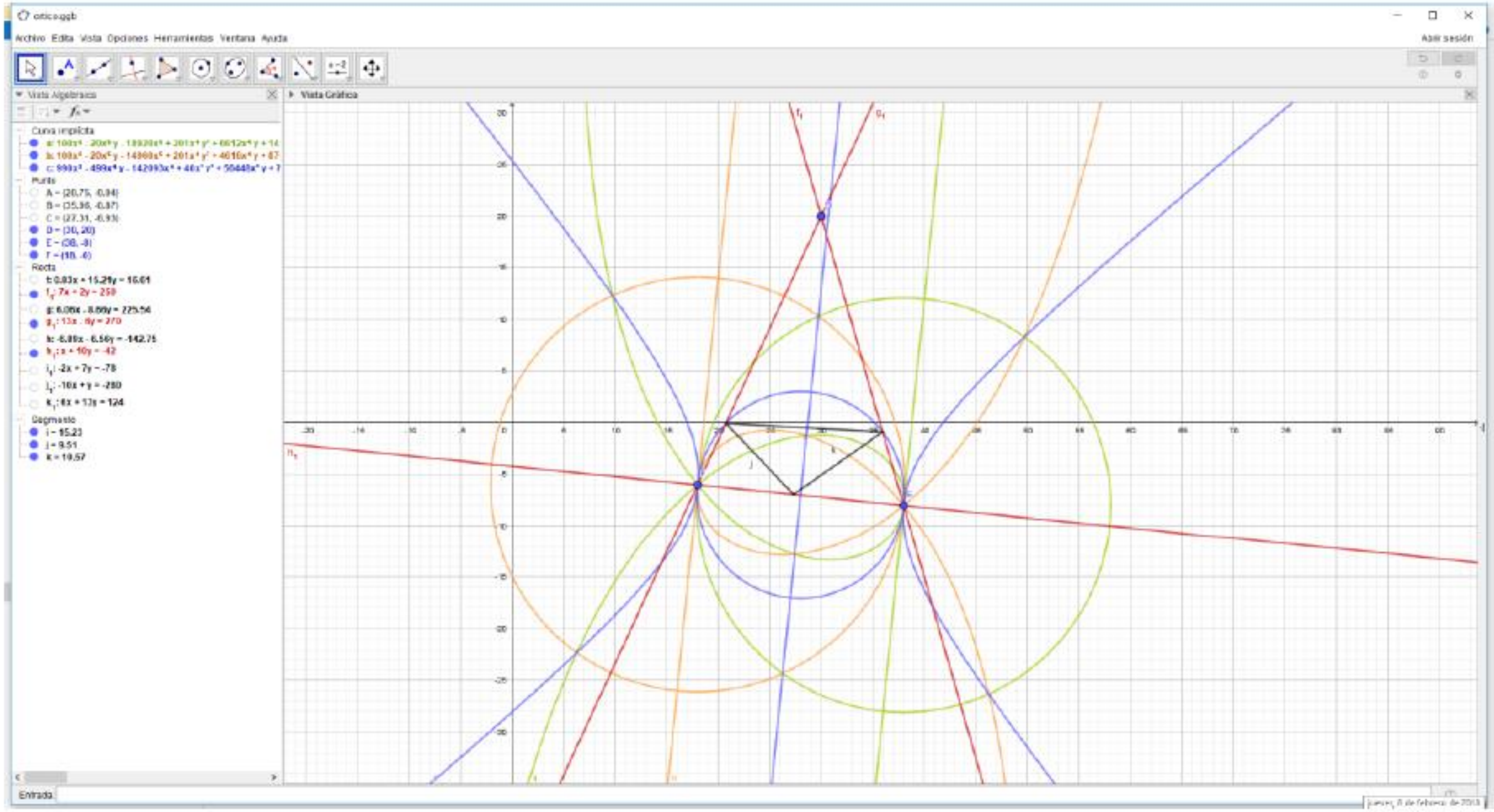
ACTIVIDAD POR DESCUBRIMIENTO:
¿Qué relación hay entre los ángulos central e inscrito de la circunferencia?

Contexto: Esta actividad está planteada para alumnos de 1º de la E.S.O, que estudian el bloque de geometría, particularmente, las relaciones entre los ángulos en la circunferencia.
Se trata de que averigüen la relación que existe entre los ángulos central e inscrito de la circunferencia.

Medio didáctico: GeoGebra y cuestionario en papel.

Instrucciones para el alumno:

- Abre en tu ordenador el archivo de GeoGebra que se te ha proporcionado.
  https://ggbm.at/QjdffSsk
- Observa los ángulos que aparecen y responde a las dos primeras preguntas del cuestionario.
- Después, mueve los puntos sobre la circunferencia para obtener diferentes valores de los ángulos y, anota 6 parejas de valores en la tabla de la pregunta 3 del cuestionario. Es aconsejable que busques valores enteros para que te sea más fácil encontrar la relación entre los ángulos.
- Responde a la pregunta 4 del cuestionario.



or ende, el triángulo órtico será equilátero cuando el punto D se encuentre en una de las intersecciones de
s curvas de color naranja, azul y verde (habrá 4 soluciones).

Final considerations

- **Learning:** helps to develop professional skills, promotes creativity,...
- **Teaching:** Teachers find difficult to understand the internal mathematical models of GGB-ART

Internal model	User model	Interaction of models
Algebraic geometry	**Euclidean geometry**	**Instrumentation**

How to design activities in order to avoid the black box effect when working with ART?

- **Training:** the use of technology in the classroom requires more training in mathematics, didactics and computer science
- **Ongoing project**

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