Investigation and Visual Explanation in Dynamic Geometry Environment

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Introduction

- Work in Dynamic Geometry Environment can encourage students to investigate in geometry.
- Students can experiment by dragging the geometrical objects they create and thus deduce properties and generalise them.
- DGE can be useful also in understanding of proofs through investigation and experimentation.
- Research question: How to utilise the program GeoGebra in geometric problem solving most efficiently?
Introduction

- Research design
- Participants in the experiment
- Problem – what is the problem in geometry?
- Problem solving process
- Three types of geometric problems and their solutions
- Findings and conclusions
Research design

- We used qualitative research design - **case studies** on the plane geometry problem solving strategies of prospective teachers in two different environments: Paper and Pencil Environment and Dynamic Geometry Environment – with the possibility to use them simultaneously.

- **Selection of participants:** We preferred communicative students interested in Euclidean geometry to facilitate the data collection process.

Participant A: 2\textsuperscript{nd} year female Bc. student, J. Selye University, Komárno
Participant B: 1\textsuperscript{st} year male Mgr. student, J. Selye University, Komárno
Participant C: 3\textsuperscript{rd} year female Bc. student, Constantine the Philosopher Univ. in Nitra
Participant D: 1\textsuperscript{st} year female PhD. student, Constantine the Philosopher Univ. in Nitra
Research design

- **Data collection** came from four sources: paper worksheets, GeoGebra files, audio records, questionnaires.
- Data collection period includes individual sessions with each participant. They subsequently received 3 worksheets with the text of one of the 3 given problems. They were allowed to work for 30 minutes on each problem.
- During the problem solving they used the worksheets, the program GeoGebra and an audio recording of their verbal thinking processes was made.
- After the problem solving they were asked to fill in a questionnaire and to write down their opinion in utilisation of GeoGebra software.
What is a Problem?

- A problem is a situation that consists of exact open questions which will **challenge somebody intellectually** who is not in immediate possession of a direct answer.
- Problem solving is a process of engaging in a task or a situation for which there is **no obvious or immediate solution**.
- In Dynamic Geometry Environment, students are able to develop alternative strategies that could not be easily explored in a Paper and Pencil Environment.
Problem solving process

- Visualisation of the problem (sketch)
- Investigation, exploration phase
- Conjecture, hypothesis creation
- Verification (optionally generalisation)
- Explanation and proof

(verification refers to the truth of a statement while explanation provides insight into why this statement is true)
What is visual explanation?

(Perhaps the most famous and certainly one of the oldest visual explanations in mathematics is the famous visual proof of the Pythagorean theorem.)

**Tufte’s principles of visual explanation** (after Casselman):

1. Tone down secondary elements of a picture: layer the figure to produce a visual hierarchy.
2. Replace coded labels in the figure by direct ones.
3. Produce emphasis by using the smallest possible effective distinctions.
4. Eliminate all unnecessary parts of a figure.
5. Use small multiples: numerous repetitions of a single figure with slight variations.
6. Make the graphics carry a story.
Three geometric problems in the case studies

- Problem 1 and Problem 2 are so-called **open-ended problems**.
- Open ended problems give students the opportunity to engage in a process which utilises exploring a situation, making conjectures, validating conjectures and proving them.
- Problem 3 is a so-called geometric **proof task**.
- In this area DGE can contribute to students’ attempts toward proof and bridge the gap between inductive explorations and deductive reasoning.
Problem 1: Geometric relationships

1. Let E be an arbitrary point on the meridian $t_c$ of a triangle ABC. The parallel line to the line AC (resp. BC) through the point E intersects the sides of the triangle at the points $F \in AB$, $I \in BC$ (resp. $G \in AB$, $H \in AC$). What relation is there between the areas of the quadrangles AEFH and GBIE?
Solution of Participant A

- As a first step she prepared a pencil sketch with an extra sketch of quadrangles.
- Secondly, she sketched the situation in GeoGebra.

She realised that the ratio of the areas of the triangles depends on the ratio of lengths CD:ED
Solution of Participant A

She intuitively supposed that D is the midpoint of FG.

After the hint that first we need to prove that IFDI=IDGI she realised the equivalence of ratios following from parallel projection:

\[ \frac{BD}{BG} = \frac{CD}{ED} \]

\[ \frac{AD}{AE} = \frac{CD}{ED} \]

‘Since the lengths of AD and BD are equal, it follows that the lengths of GD and FD are equal too’
Solution of Participant B

1. Let $E$ be an arbitrary point on the meridian $t_c$ of a triangle $ABC$. The parallel line to the line $AC$ (resp. $BC$) through the point $E$ intersects the sides of the triangle at the points $F \in AB$, $I \in BC$ (resp. $G \in AB$, $H \in AC$). What relation is there between the areas of the quadrangles $AEFH$ and $GBIE$?

He sketched immediately two different positions for point $E$ and noticed the key role of the remaining triangles.
Solution of Participant B

He began the solution with extreme cases:

\[
E = C
\]

\[
\triangle FGE \sim \triangle ABC
\]

\[
\Rightarrow \triangle FED = \triangle DGE
\]

\[
\angle HEIC \Rightarrow
\]

\[
\angle HEC = EIC
\]
Missing step: Why IFDI=IDGI?

The student supposed that intuitively.
Solution of Participant D

Unnecessary step and Incorrect implication

\[ \frac{|AH|}{|HC|} = \frac{|B|}{|IC|} \]

\[ AF = CB \]
Solution of Participant C

She preferred paper and pencil environment. GeoGebra was used only to see the relationship between areas.

Unsuccessful solution based on comparing the angles
A short solution

- Triangles ADC and FDE, and triangles BDC and GDE are similar with the same ratio of similarity given by the position of the point E on DC.
- The triangles ADC and BDC are equilateral, so triangles FDE and GDE are also equilateral.
- The areas of given quadrangles are differences of equal areas, therefore they are equal.
Problem 2: Locus of points

2. What is the locus of the orthocentres of such triangles ABC, where side AB has constant length $c$ and also the size of angle $ACB = \gamma$ is constant.
Solution of Participant A

To see the locus of orthocentres in GeoGebra gave her a hint that the key idea of the solution is about inscribed angles.

She didn’t investigate all possibilities for point C on the circle.
Solution of Participant B

He collected the pencil sketch after using the locus function of GeoGebra.

He was not able to explain why it is a circle.

Hint: See the angles in quadrangle BMEC
Solution of Participant D

She was the only one who found the locus correctly
A short solution

Perpendiculars to the sides AC and BC of $\gamma$ form the same angle $\gamma$. 
Problem 3: Proof

3. In an acute triangle ABC, the altitude AD is given. The angle bisector of angle BAD intersects the side BC at the point E and the angle bisector of angle CAD intersects the side BC at the point F. The circumcircle of the triangle AEF intersects the sides AB and AC at the points G and H, respectively. Prove that the lines EH, FG and AD meet at one point.
Solution of Participant A

3. In an acute triangle ABC, the altitude AD is given. The angle bisector of angle BAD intersects the side BC at the point E and the angle bisector of angle CAD intersects the side BC at the point F. Circumcircle of the triangle AEF intersects the sides AB and AC at the points G and H, respectively. Prove that the lines EH, FG and AD meet at one point.

After an unsuccessful sketch she used only GeoGebra.

She was given a hint to observe the triangle AEF.
Solution of Participant B

He was the only one who realised that the sentence is valid also if there is an obtuse angle at the vertex A.
After an unsuccessful sketch she used only GeoGebra.

She was the only one who found that the three lines are supposedly altitudes in the triangle AEF. To prove that, she tried to use inscribed angles.
An acute triangle ABC with the altitude AD is given. The axes of angles BAD and CAD intersect the side BC in the points E and F. The circumcircle of the triangle AEF intersects the sides AB and AC in the points G and H. Prove that the lines EH, FG and AD intersect at one point.
An acute triangle $ABC$ with the altitude $AD$ is given. The axes of angles $BAD$ and $CAD$ intersect the side $BC$ in the points $E$ and $F$. The circumcircle of the triangle $AEF$ intersects the sides $AB$ and $AC$ in the points $G$ and $H$. Prove that the lines $EH$, $FG$ and $AD$ intersect at one point.
The inscribed angles belonging to arc AE are equal.
Visual explanation - step 2

The angle at A is divided by axis into two equal parts.
Visual explanation - step 2

Triangles ADF and AIH are similar.
Visual explanation - step 2

Lines AF and HE are perpendicular.
Visual explanation - step 3

The same consideration for the lower side gives that at point J is also a right angle.
Thus HE, FG and AD are 3 altitudes in triangle AEF, and the point K is the orthocentre of the triangle.
1. In which phase of problem solving was GeoGebra the most helpful:

- A) data sketching and a better understanding of the task
- B) to see the relationships
- C) to formulate generalisations (possibility to observe multiple cases with movement)
- D) to prove the discovered relationships
Questionnaire

2. What type of geometric task did you find most beneficial to use GeoGebra?

- A) discovery of relationships between shapes or parts of shapes
- B) to determine the locus of points
- C) proofs
Findings

- Three of four participants marked that GeoGebra was the most helpful to see the relationships and to formulate generalisations mainly by using the possibility of moving or dragging the objects.

- The participants picked up different types of tasks when they found more beneficial to use GeoGebra – two of them chose the possibility A) discovery of relationships between shapes or parts of shapes.

- Three participants (as prospective math teachers) emphasised in their answers the usefulness of GeoGebra not only in learning but also in teaching geometry. They appreciated also the visual explanation possibilities in the program.
Conclusions

- Our case studies show that prospective mathematics teachers used GeoGebra program to solve geometric problems from the beginning rather than paper and pencil.
- They were able to see more relationships by dragging the objects.
- Solving difficult tasks brings more challenge and more possibility to an investigation than solving simple tasks. Boring tasks do not motivate.
- GeoGebra offers a great opportunity to create appropriate visual hints during the problem solving process and visual explanation for simple overview of the solution.
References

Thanks for your attention