max planck institut
informatik

Fast Cube Tests

for LIA constraint solving

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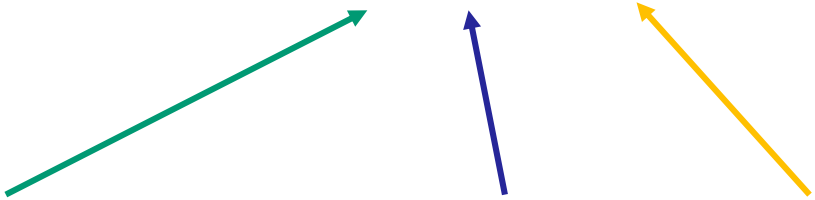
Linear Arithmetic / Linear Programming

Input: $\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i \quad \text{for } i = 1, \ldots, m$



coefficients ($\in \mathbb{Z}$) variables bound ($\in \mathbb{Z}$)

Linear Arithmetic / Linear Programming

Input: $a_i^T \mathbf{x} \leq b_i \quad \text{for } i = 1, \ldots, m$

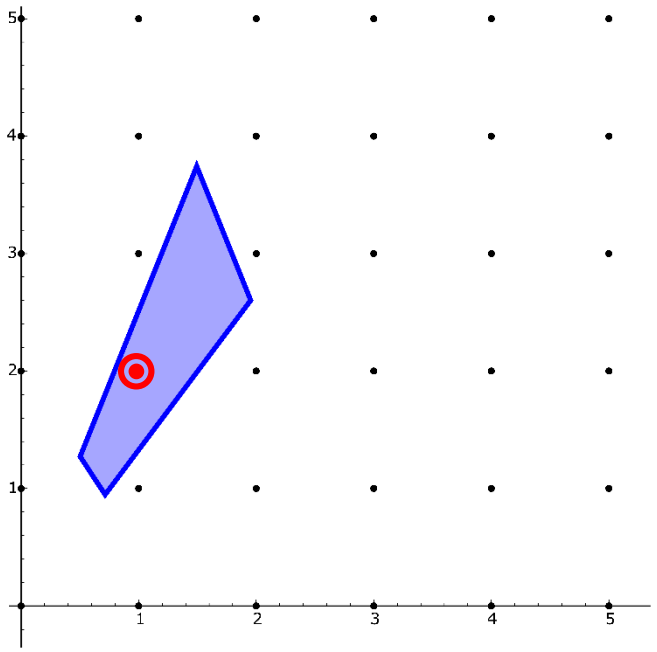
Goal: $\mathbf{x} \in \mathbb{R}^n \text{ (LRA/LP)}$ or $\mathbf{x} \in \mathbb{Z}^n \text{ (LIA/ILP)}$

Complexity: P NP

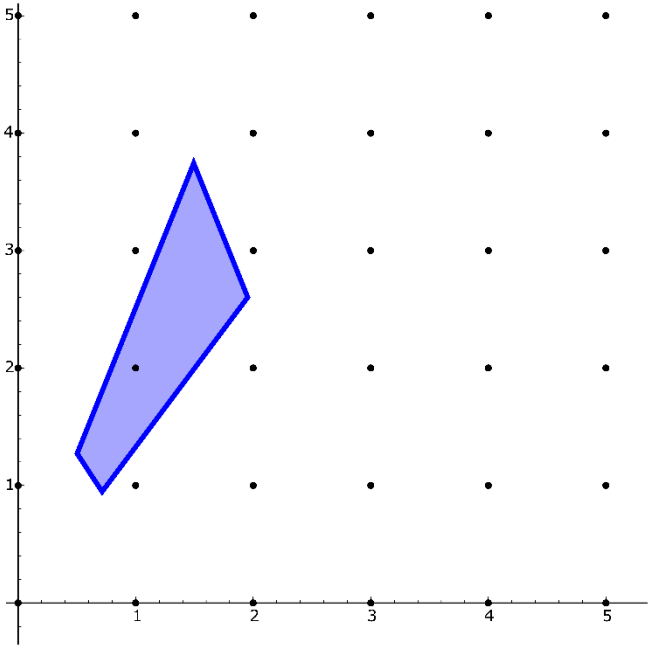
Example:

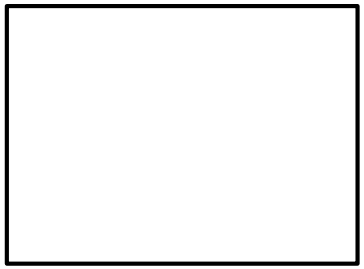
$$2y \leq 5x, \qquad 3y \geq 4x,$$
$$2y \leq -5x + 15, \qquad 2y \geq -3x + 4,$$

LIA: $(x, y) = (1, 2)$



Branch-and-Bound





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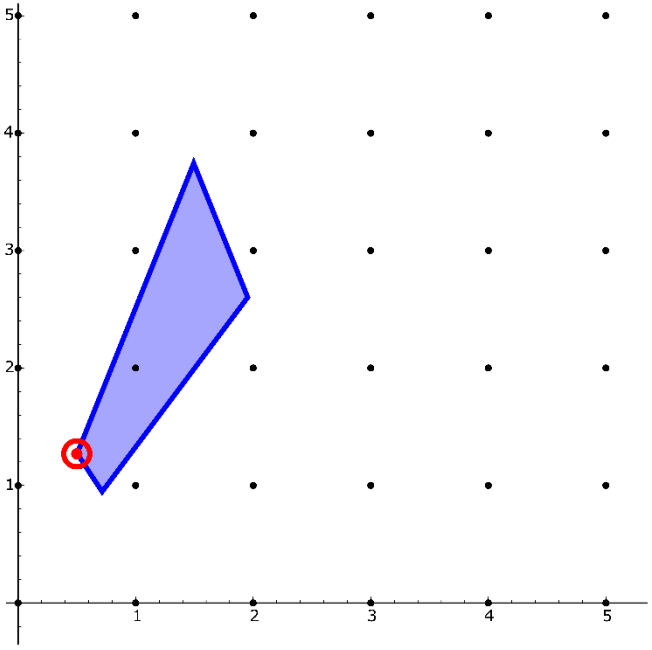

Solution:

$(x, y) = (\frac{1}{2}, \frac{5}{4})$

$$2y \leq 5x, \qquad 3y \geq 4x,$$
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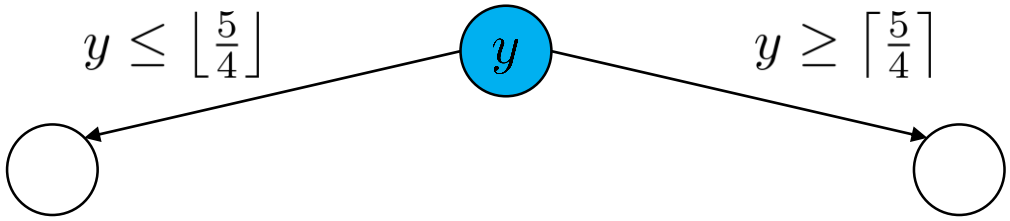
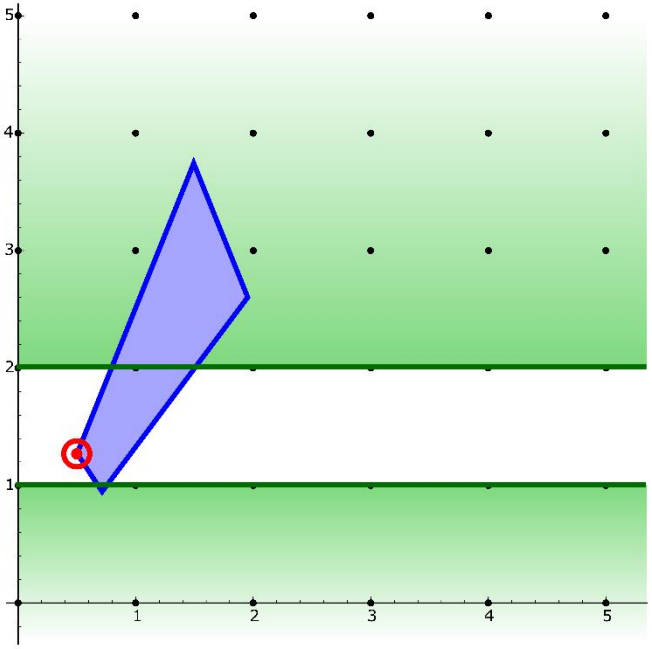
Branch-and-Bound





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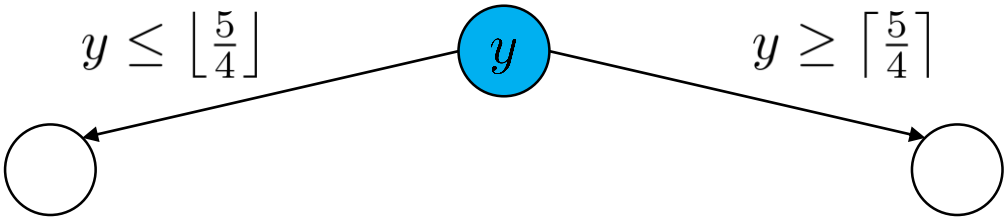

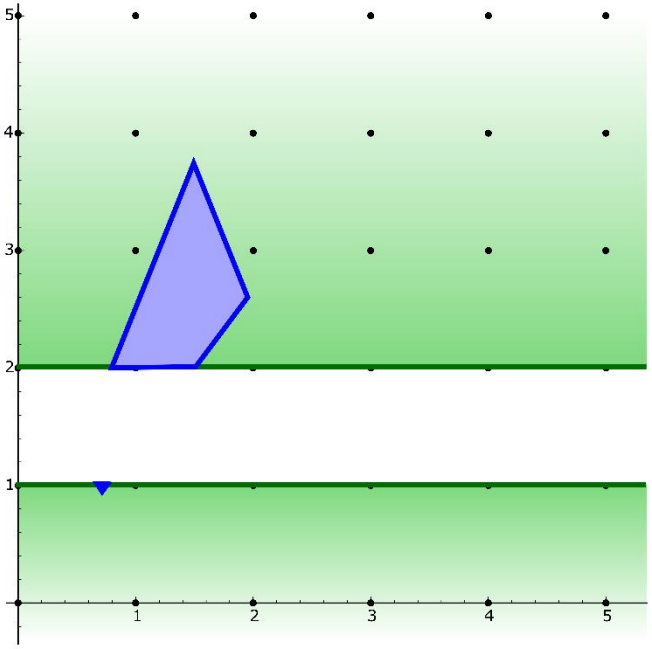

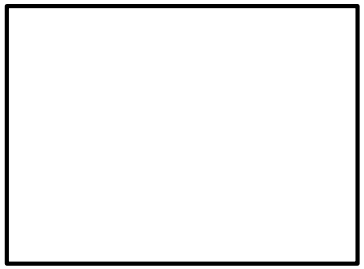

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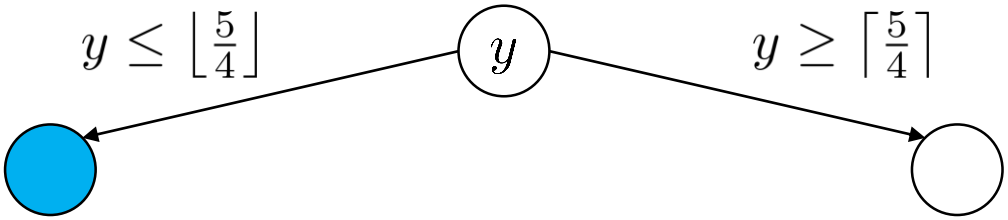

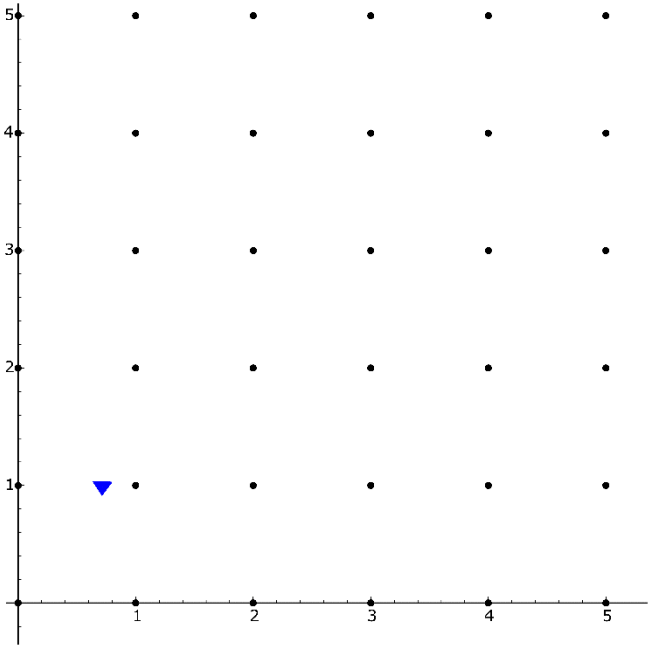

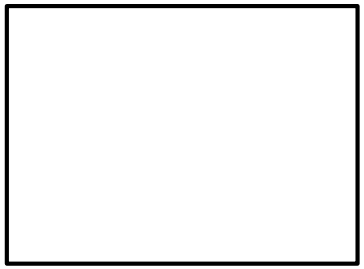

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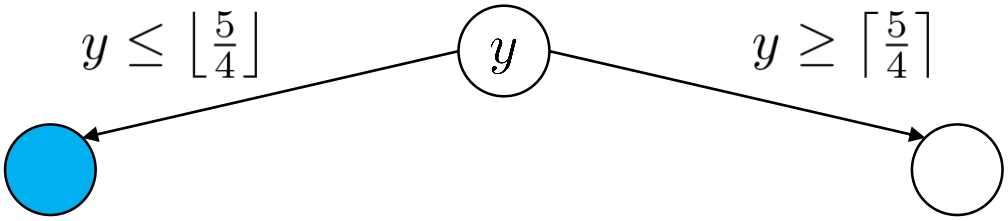

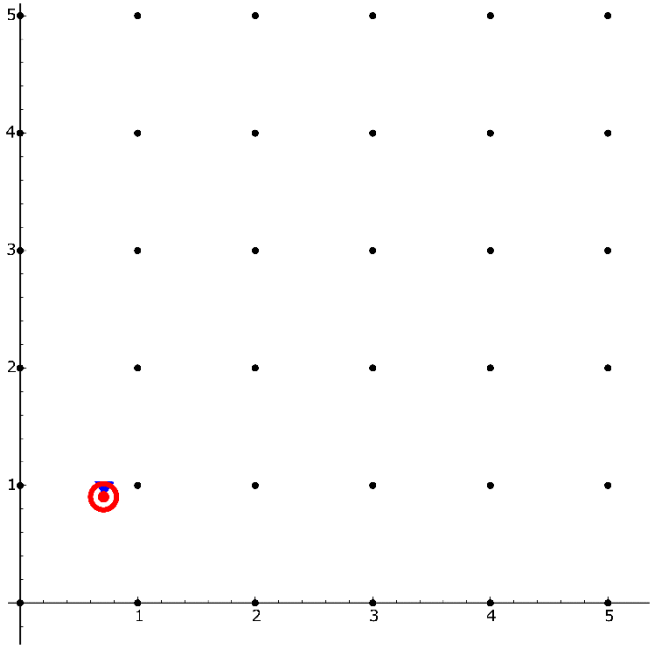

Solution:

$(x, y) = \left(\frac{12}{17}, \frac{16}{17}\right)$

$$2y \leq 5x, \qquad 3y \geq 4x,$$
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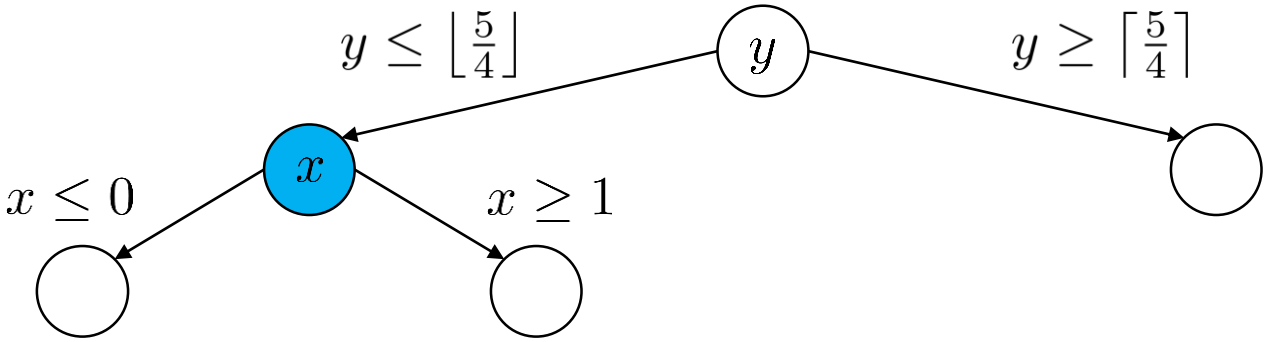
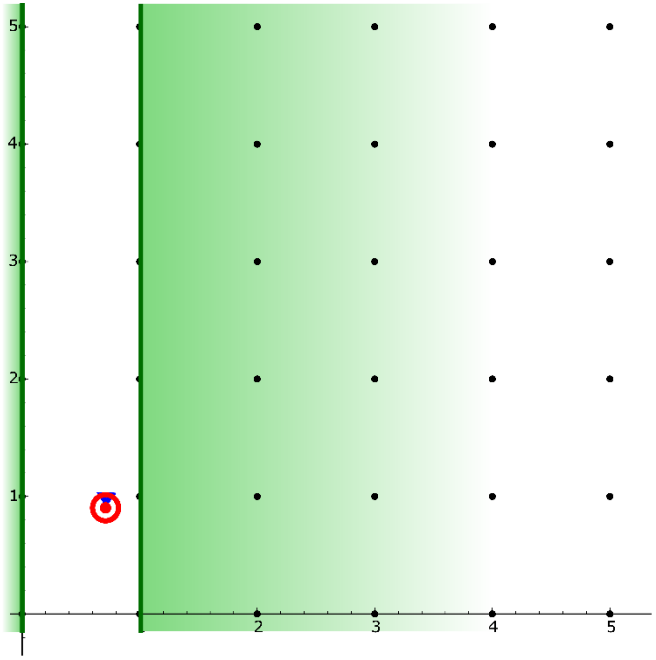
Branch-and-Bound





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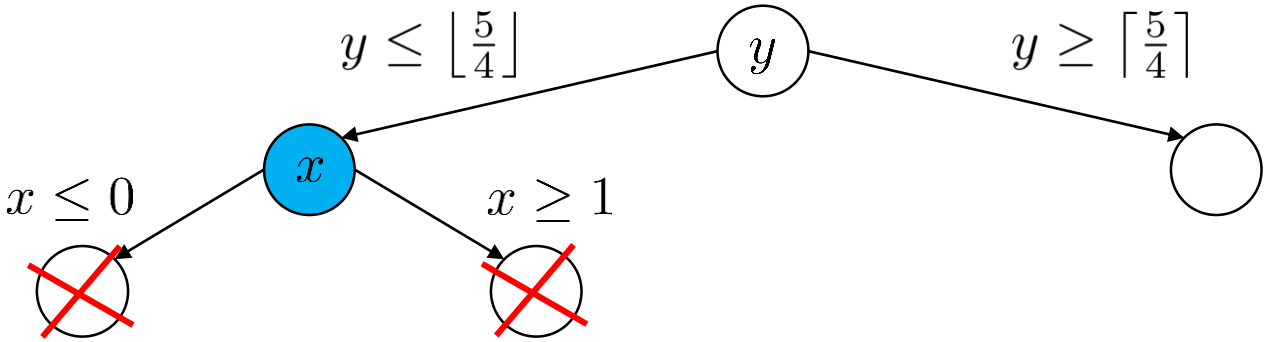

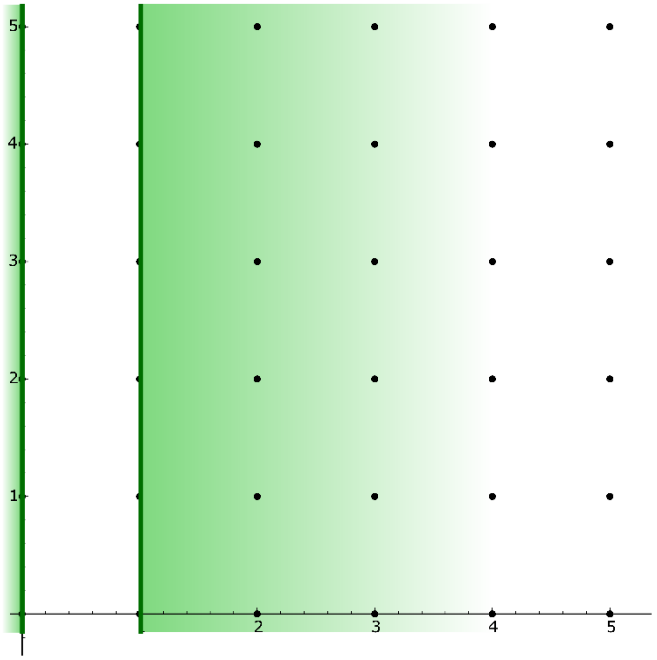

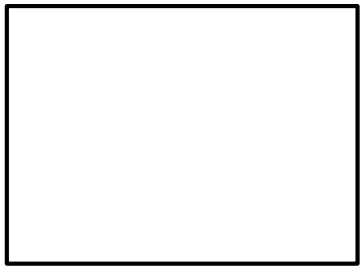

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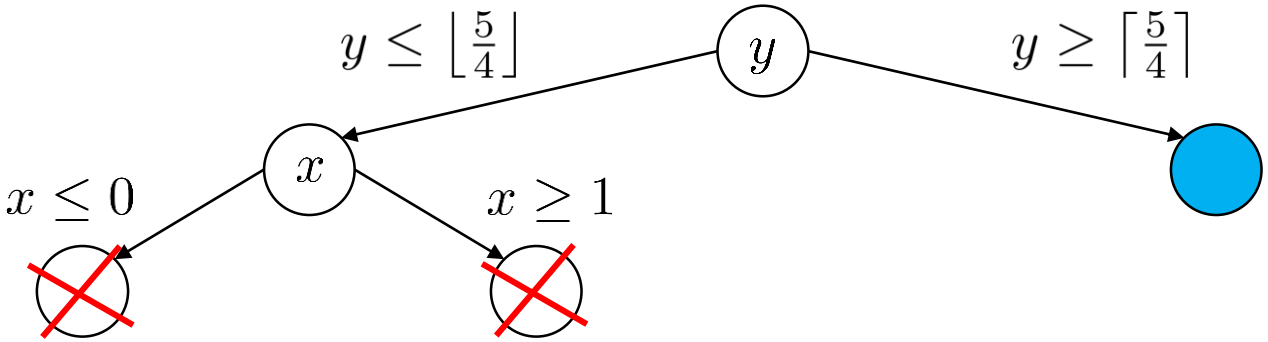
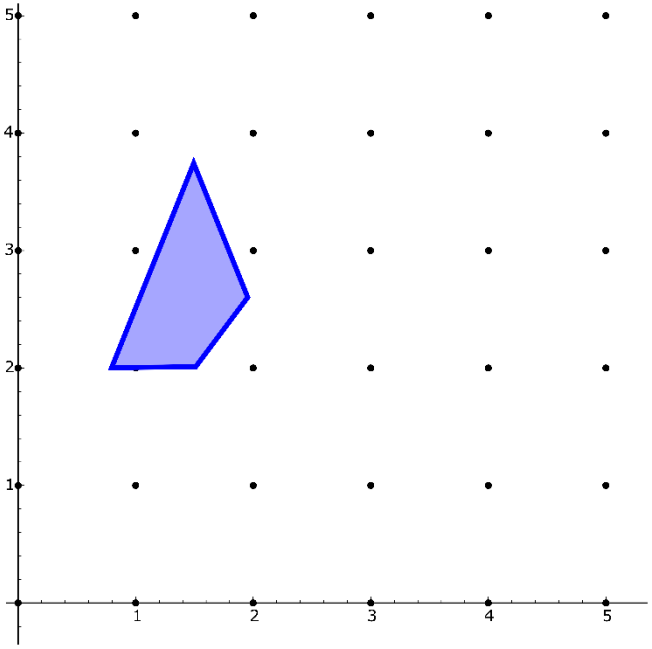
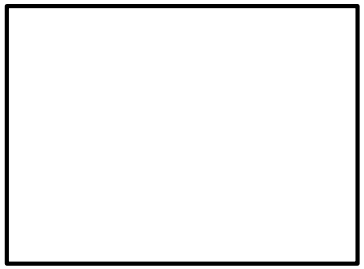
Branch-and-Bound







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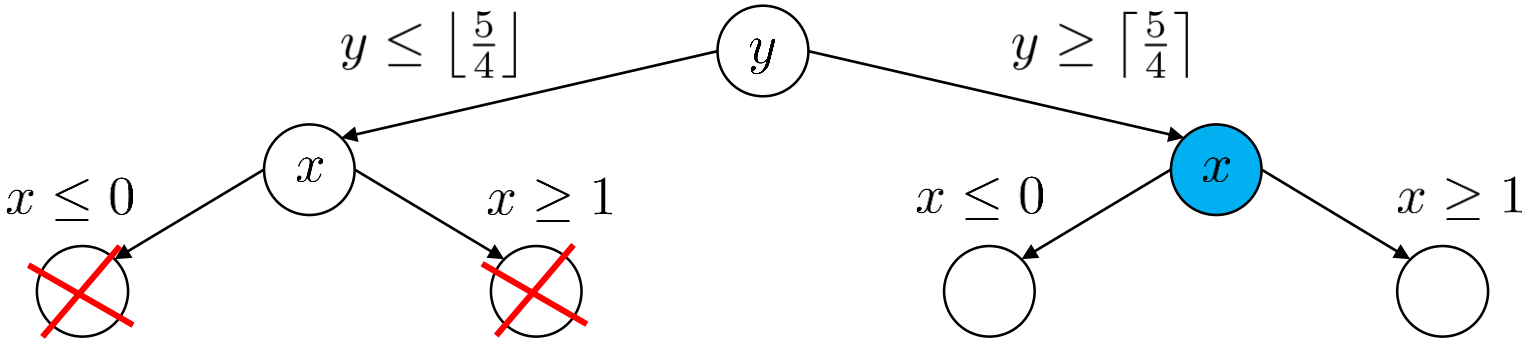
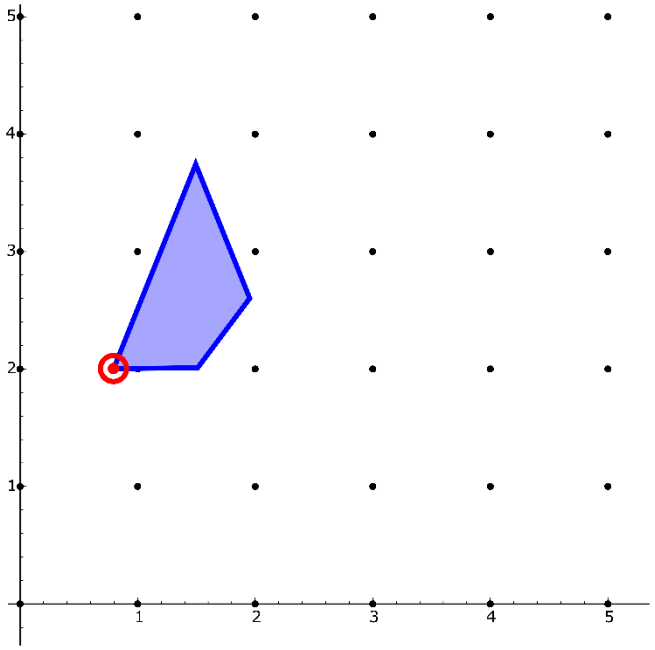


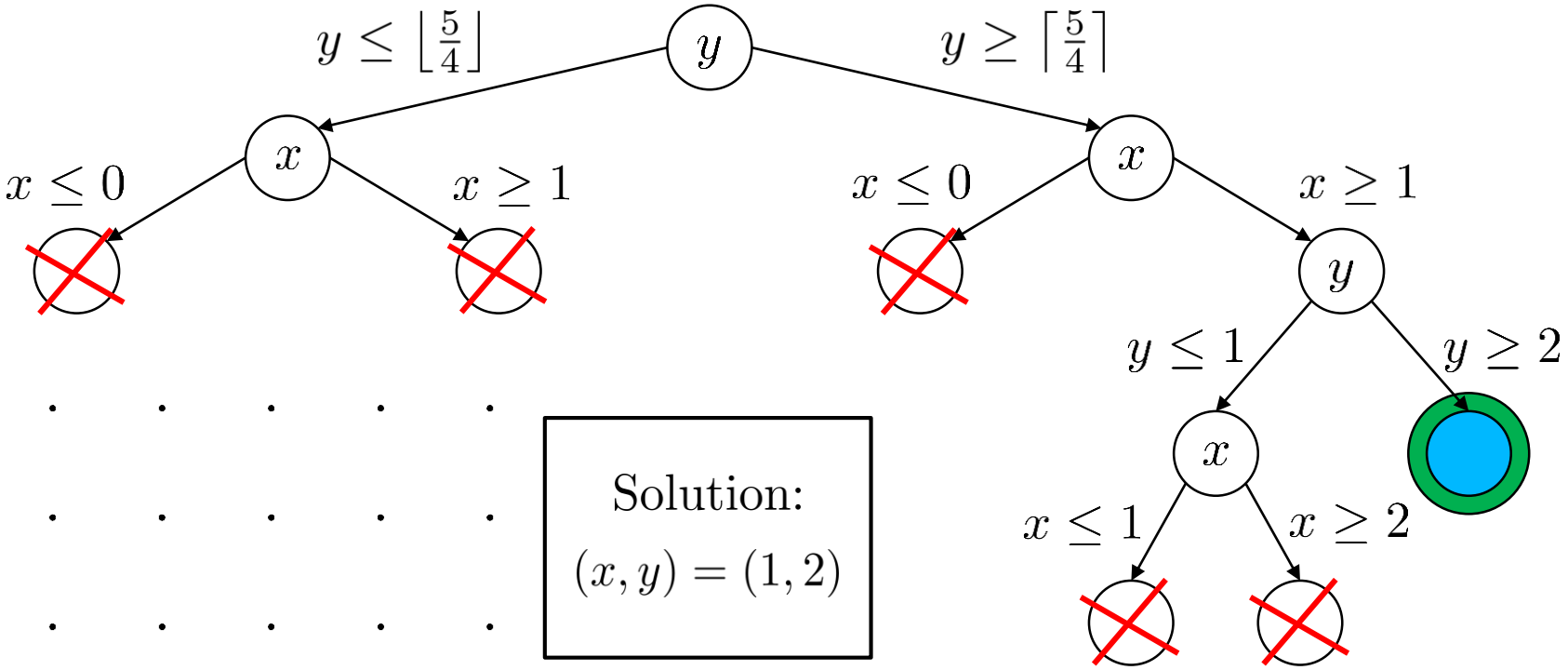


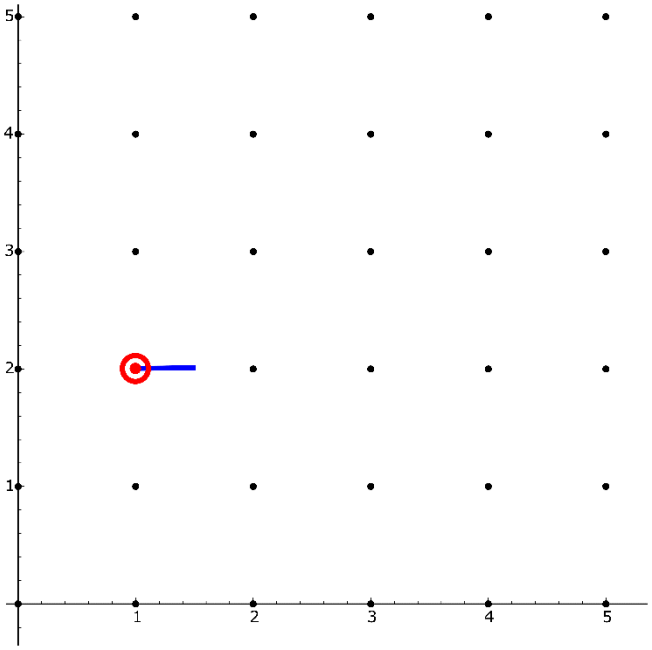
Solution:

$(x, y) = \left(\frac{4}{5}, 2\right)$

$$2y \leq 5x, \qquad 3y \geq 4x,$$
$$2y \leq -5x + 15, \qquad 2y \geq -3x + 4,$$

Branch-and-Bound





Solution:

$(x, y) = (1, 2)$

$$2y \leq 5x, \qquad 3y \geq 4x,$$
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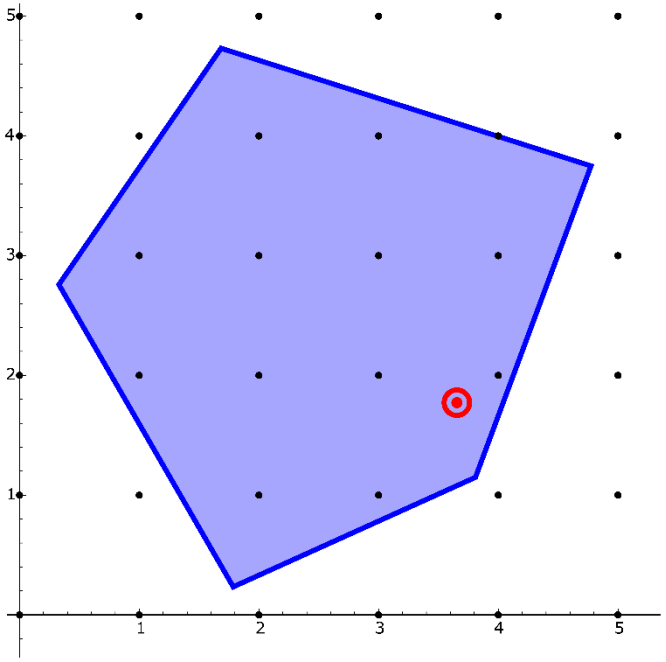
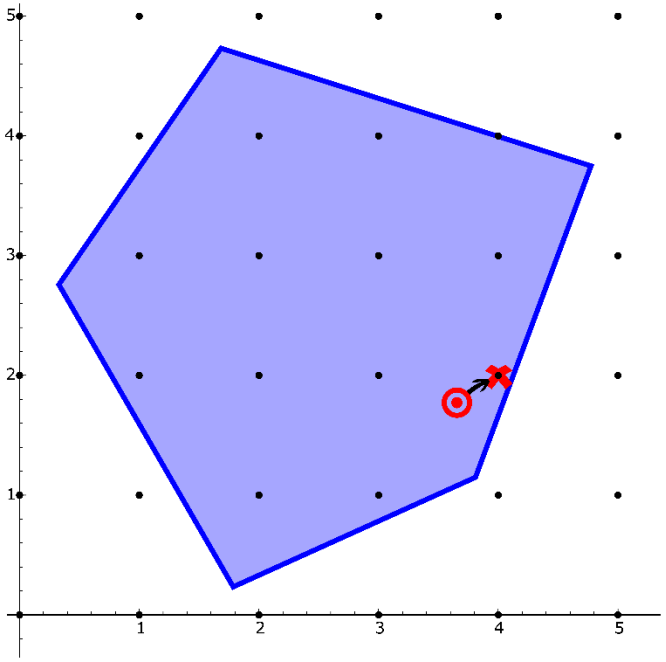
Incomplete Tests



Simple Rounding

- round real solution to closest integer point

Incomplete Tests



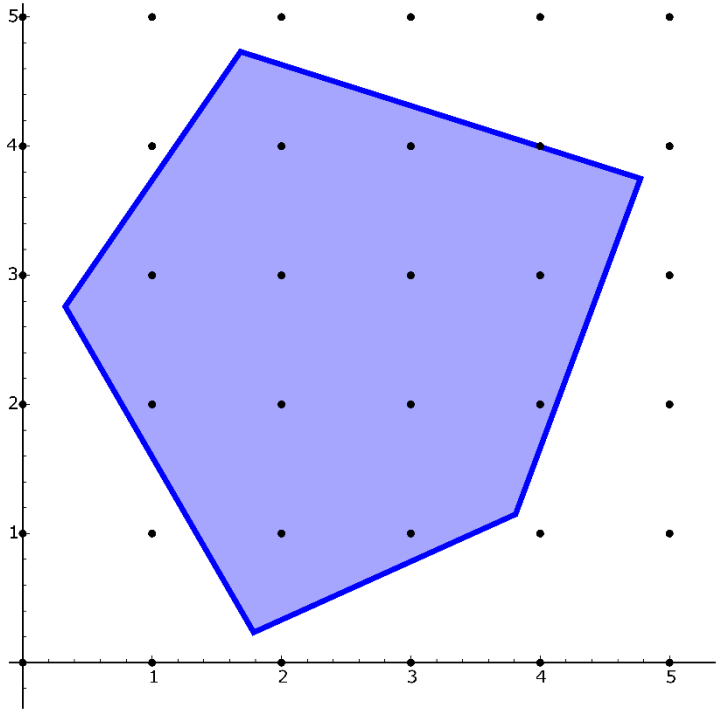
Simple Rounding

- round real solution to closest integer point

$$\lceil x \rfloor := \begin{cases} \lfloor x \rfloor & \text{if } x - \lfloor x \rfloor < \frac{1}{2} \\ \lceil x \rceil & \text{if } x - \lfloor x \rfloor \geq \frac{1}{2} \end{cases}$$

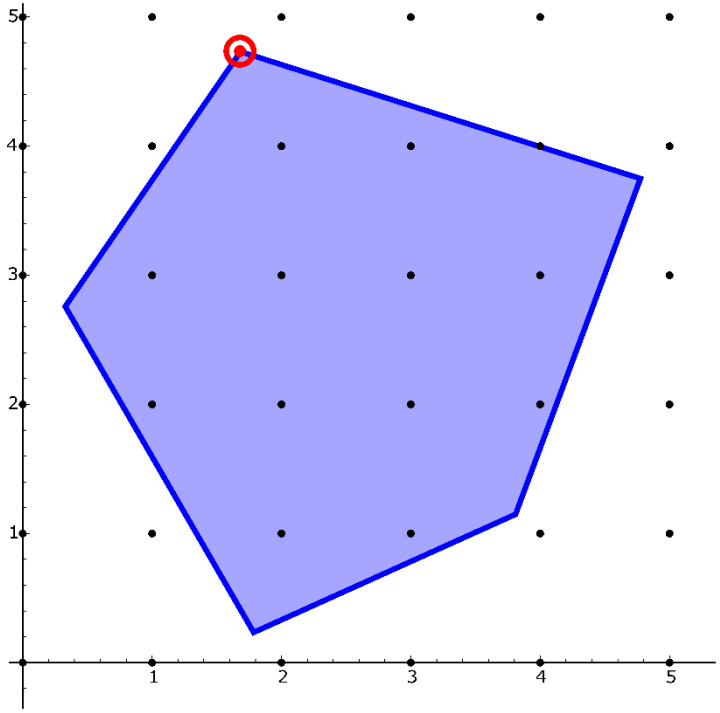
$$\text{round}((x_1, \ldots, x_n)) = (\lceil x_1 \rfloor, \ldots, \lceil x_n \rfloor)$$

Vertices



- bad for rounding

Vertices



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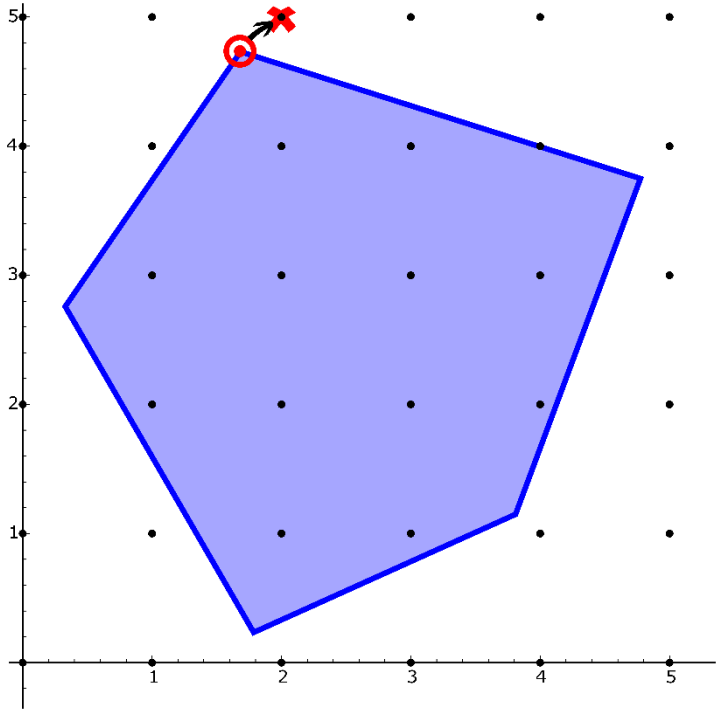

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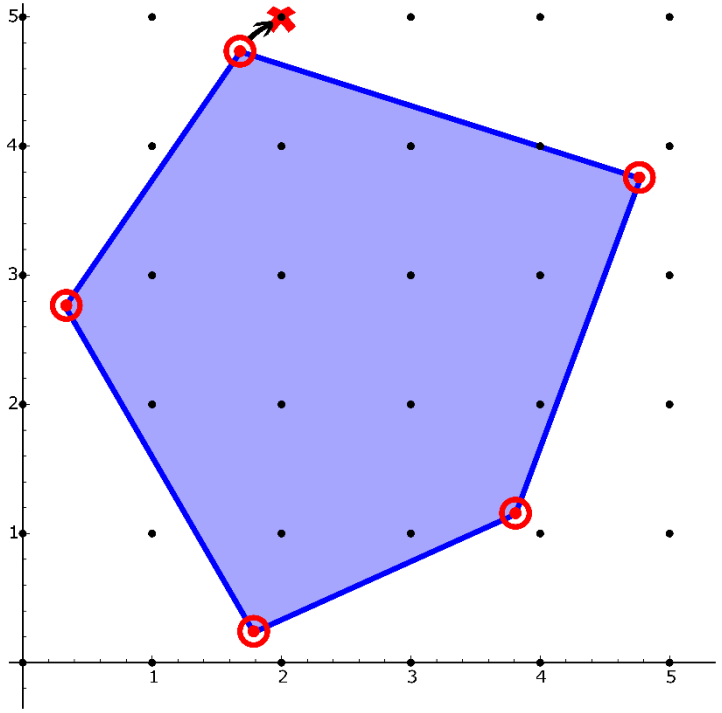

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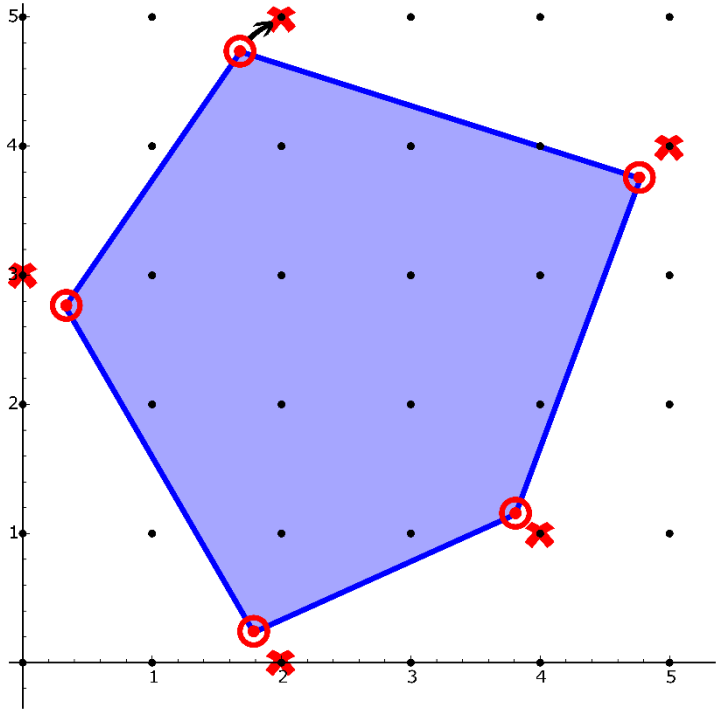
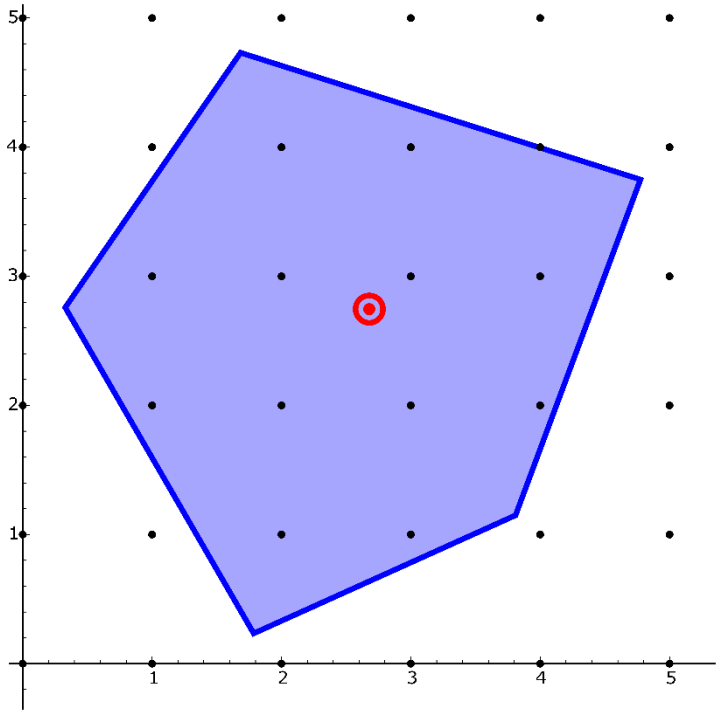


- bad for rounding
- typical output of Simplex
- SMT solvers rely on Simplex

Interior/Center Points



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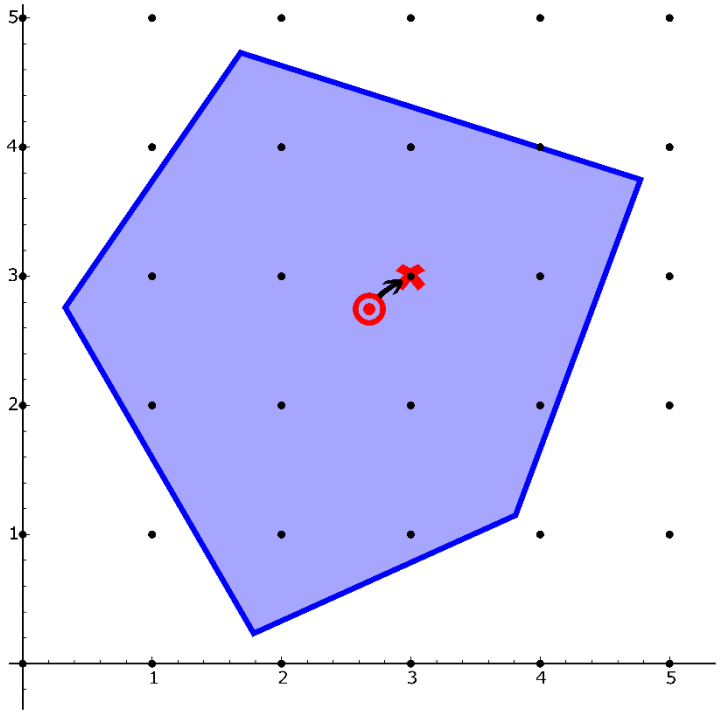
Interior/Center Points



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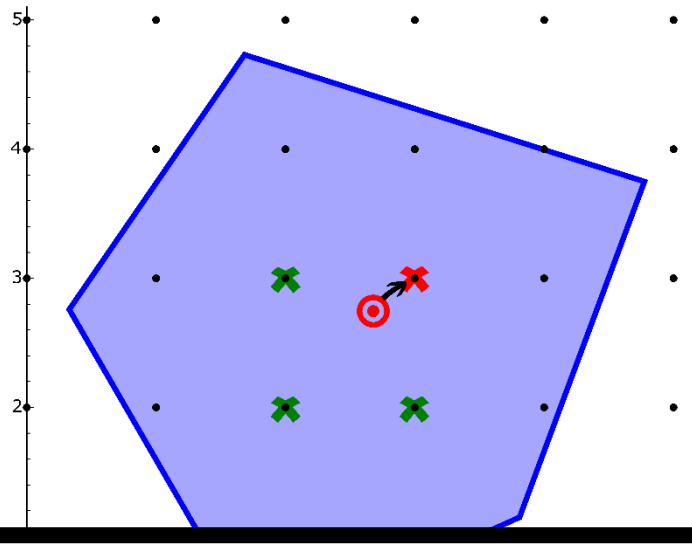
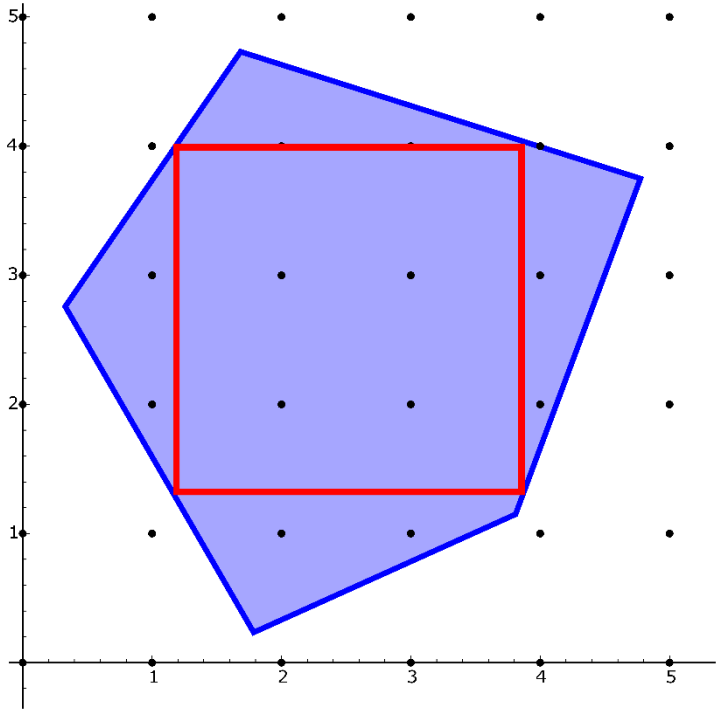
Interior/Center Points



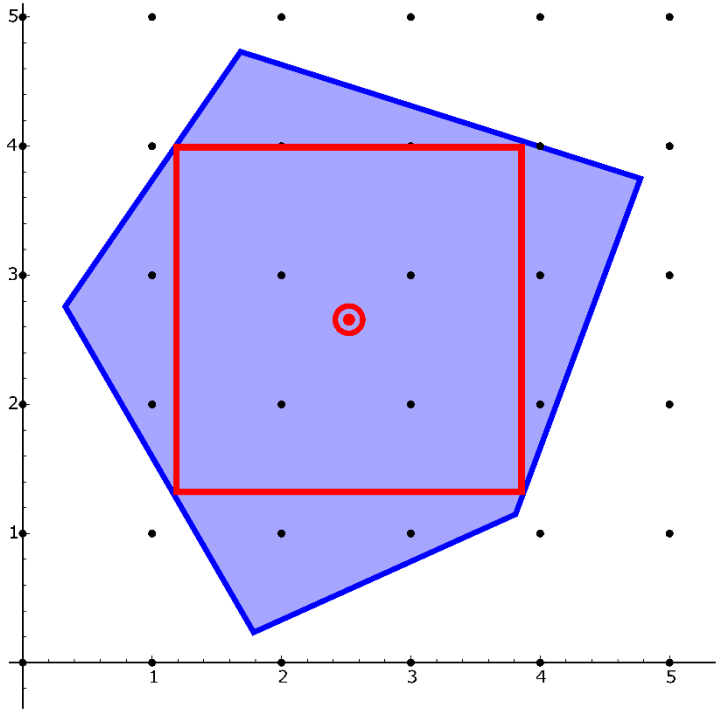
Fast Cube Tests!

Largest Cube Test



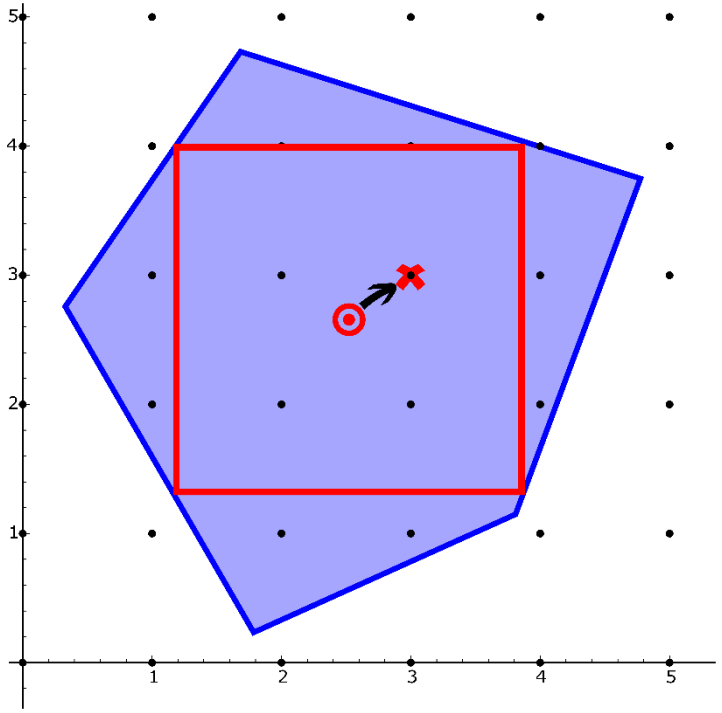
- largest cube inside the set of real solutions

Largest Cube Test



- largest cube inside the set of real solutions

Largest Cube Test



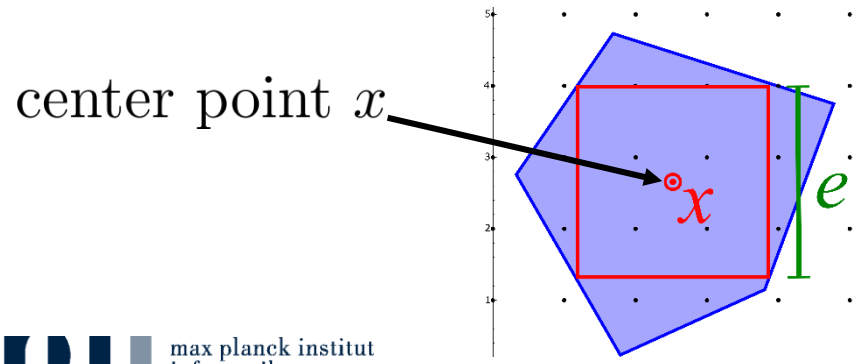
- largest cube inside the set of real solutions
- optimization LP (LRA)
- (only cubes parallel to the axes)

Largest Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$



Largest Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

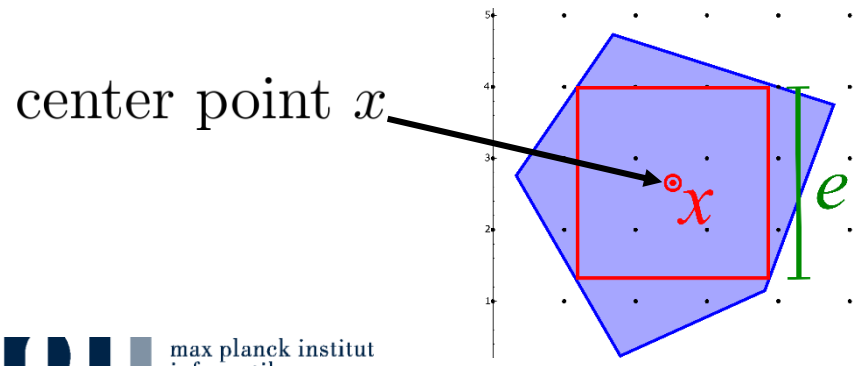
maximize e

subject to:

$$a_i^T x + \|a_i\|_1 \frac{e}{2} \leq b_i$$

for $i = 1, \ldots, m$

where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$



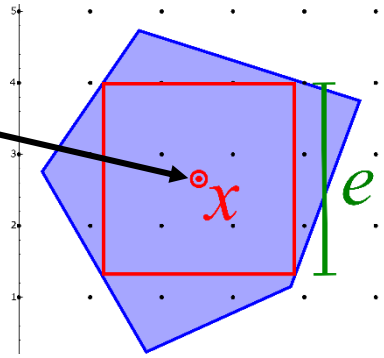
Largest Cube Test

subject to:

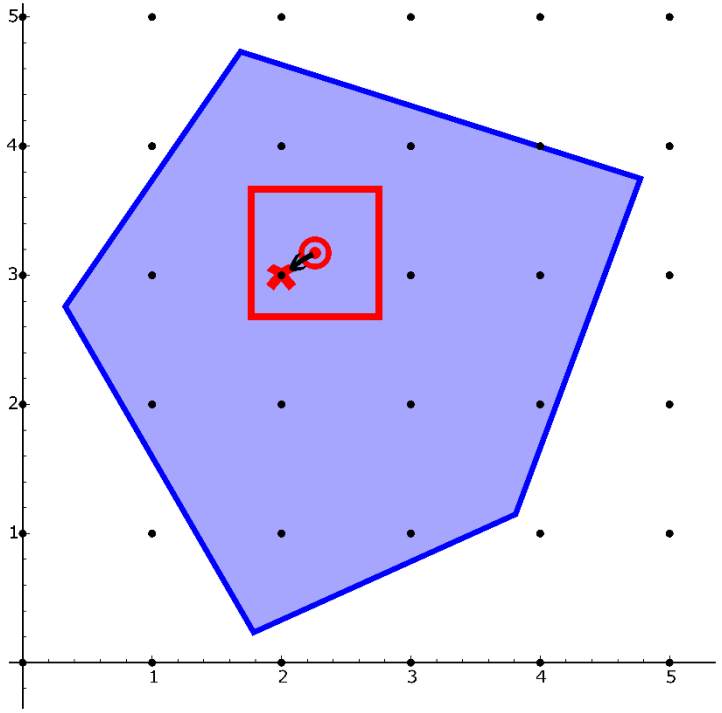
maximize e
subject to:

Without Optimization?

center point x

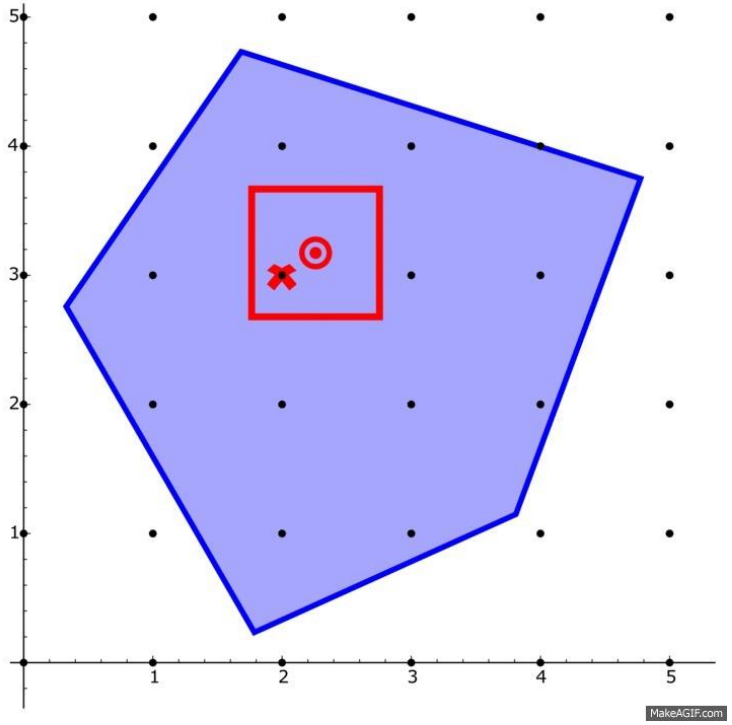


Unit Cube Test



- fix the edge length to 1

Unit Cube Test



- fix the edge length to 1
- unit cube => guaranteed integer solution

Unit Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

maximize e

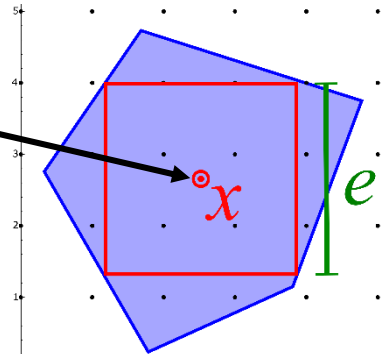
subject to:

$$a_i^T x + \|a_i\|_1 \frac{e}{2} \leq b_i$$

for $i = 1, \ldots, m$

where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$

center point x



$$e = 1$$

Unit Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

maximize 1

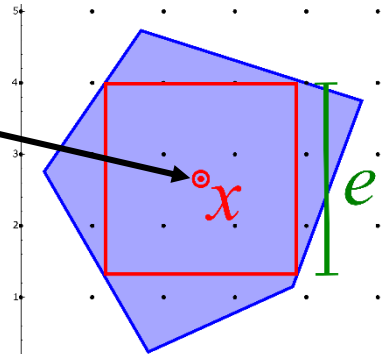
subject to:

$$a_i^T x + \|a_i\|_1 \cdot \tfrac{1}{2} \leq b_i$$

for $i = 1, \ldots, m$

where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$

center point x



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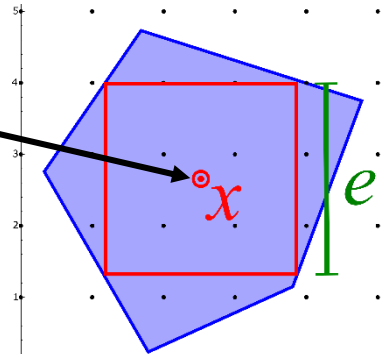

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Unit Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

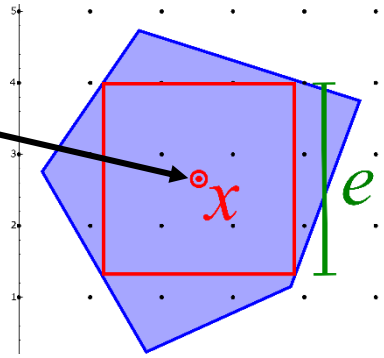
subject to:

$$a_i^T x + \|a_i\|_1 \cdot \frac{1}{2} \leq b_i$$

for $i = 1, \ldots, m$

where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$

center point x



$$e = 1$$

Unit Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

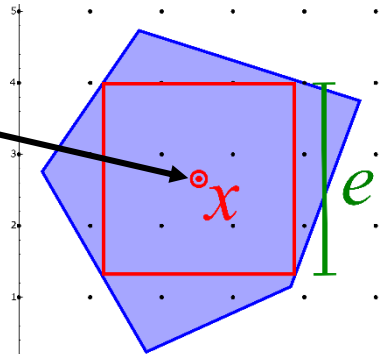
subject to:

$$a_i^T x \leq b_i - \|a_i\|_1 \cdot \frac{1}{2}$$

for $i = 1, \ldots, m$

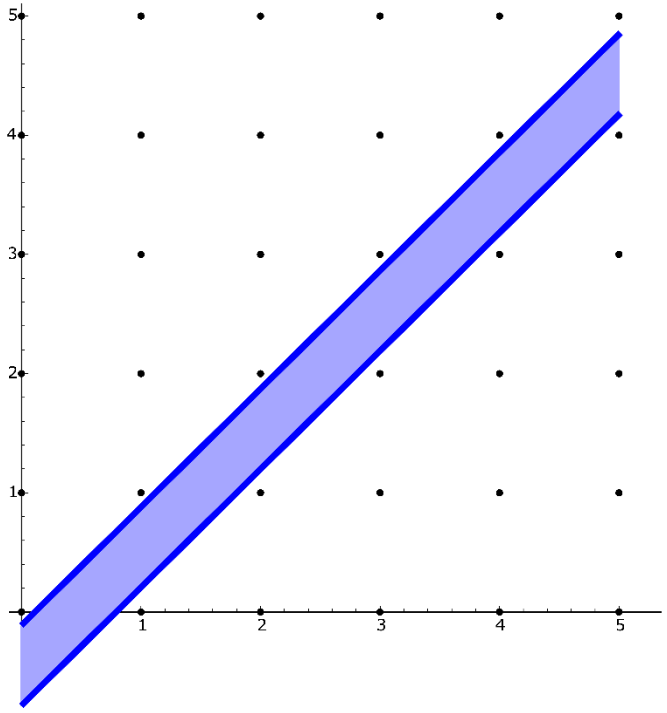
where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$

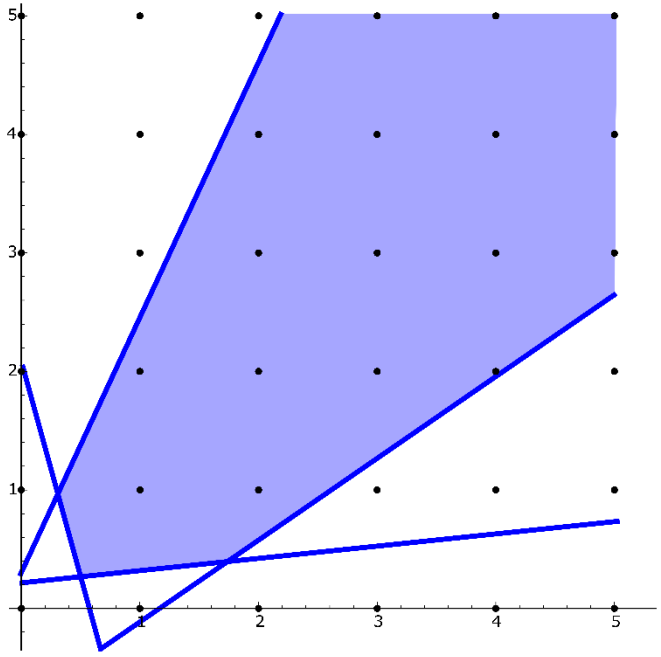
center point x



$$e = 1$$

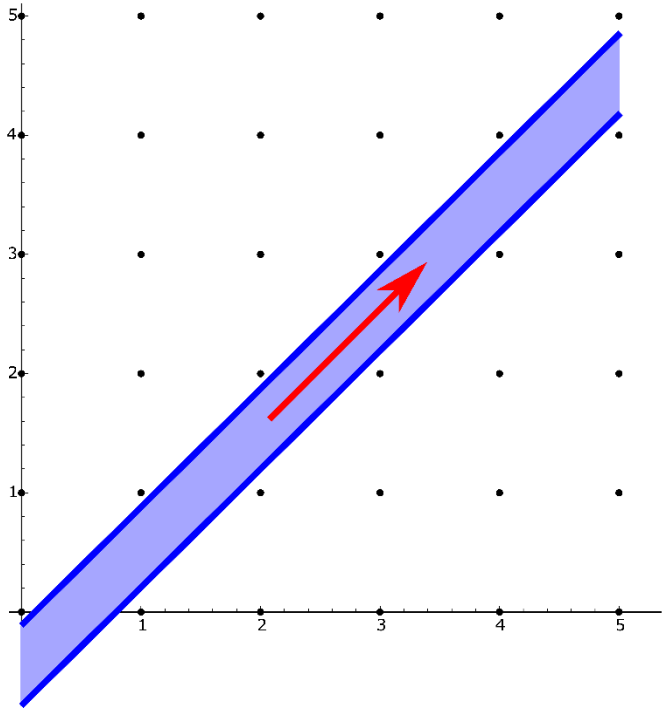
Unbounded Problems

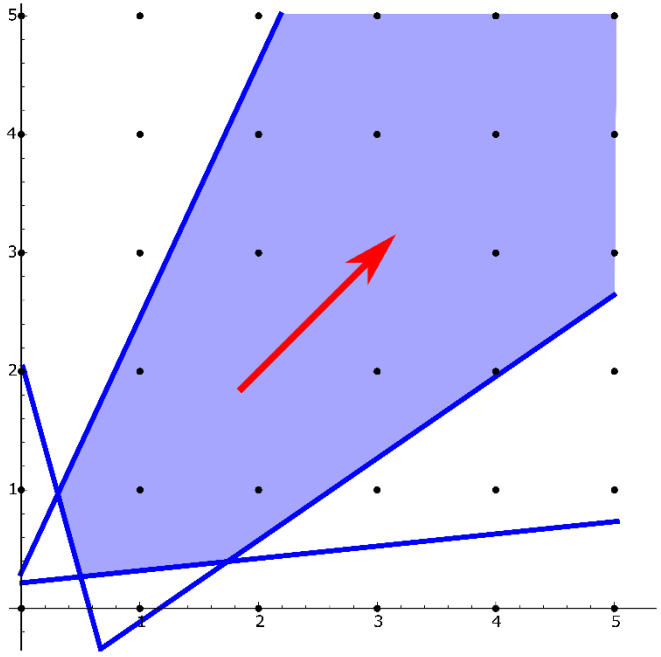




Requirement: unbounded direction

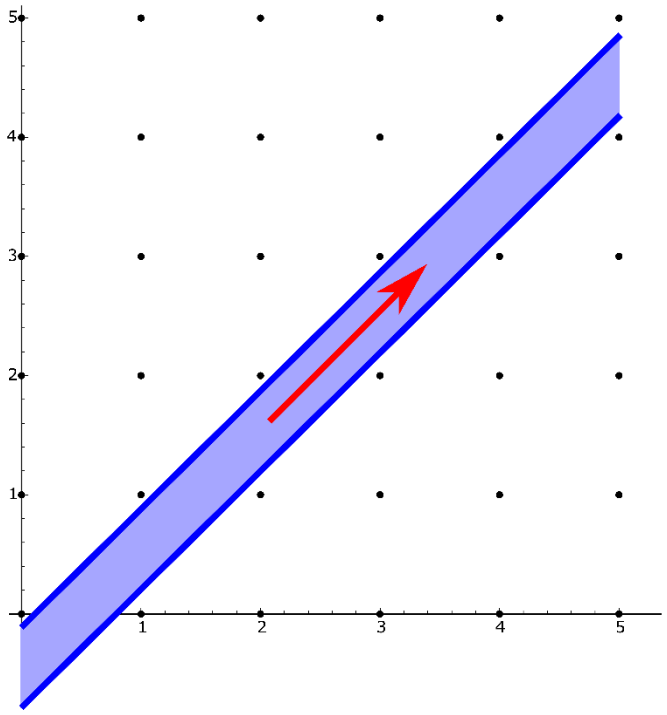
Unbounded Problems

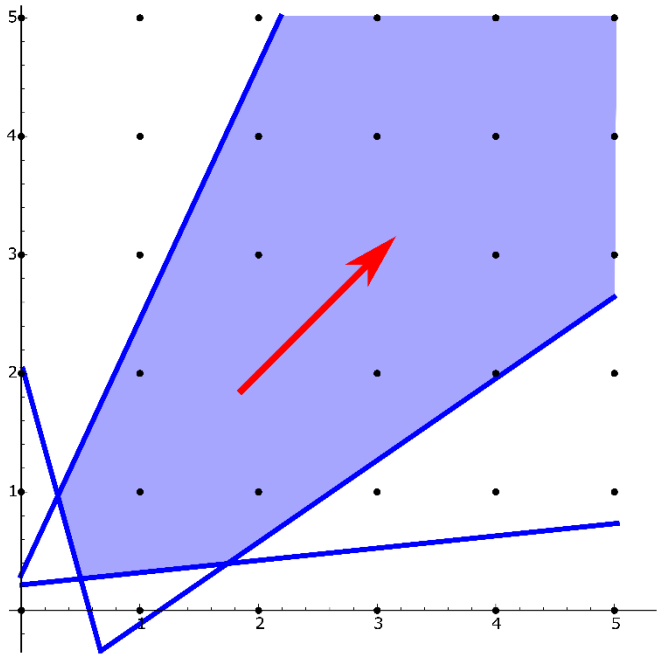




Requirement: unbounded direction

Unbounded Problems

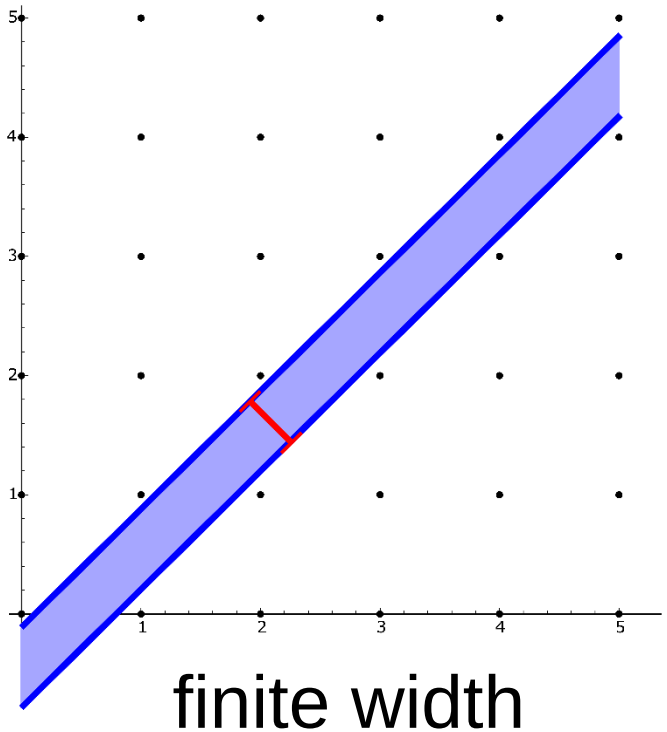




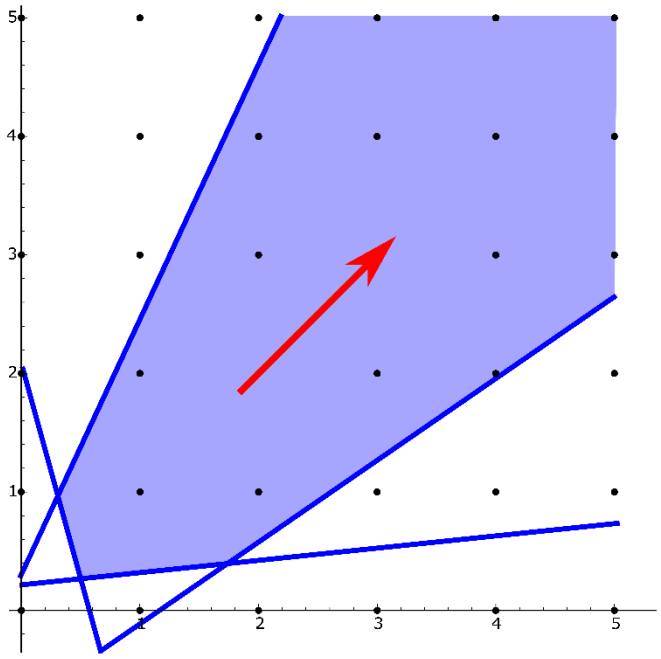
Infinite Width: all directions unbounded

$$\forall c \in \mathbb{R}^n. \quad \min c^T x = -\infty \quad \text{or} \quad \max c^T x = \infty$$

Unbounded Problems



finite width



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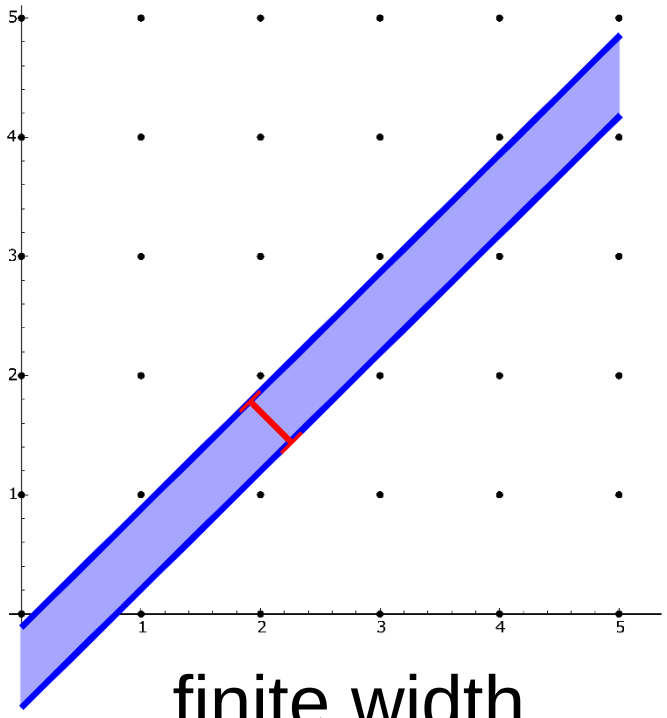
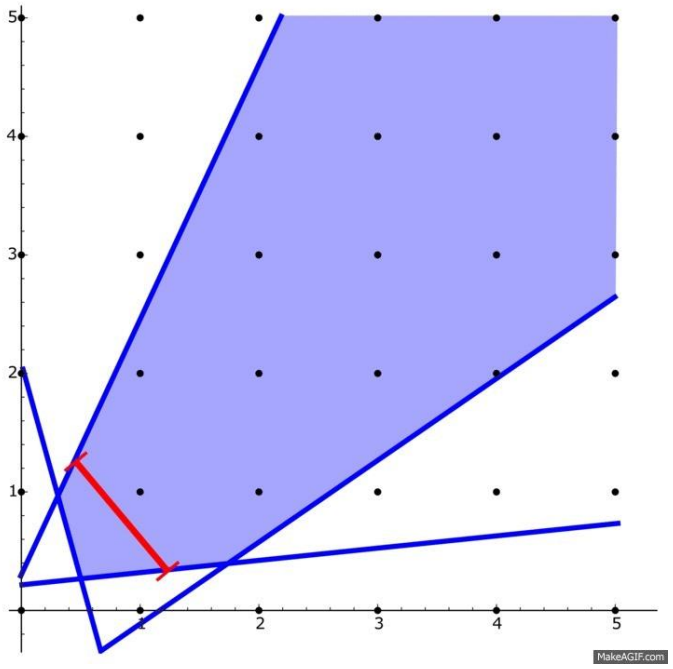
Unbounded Problems



finite width



infinite width

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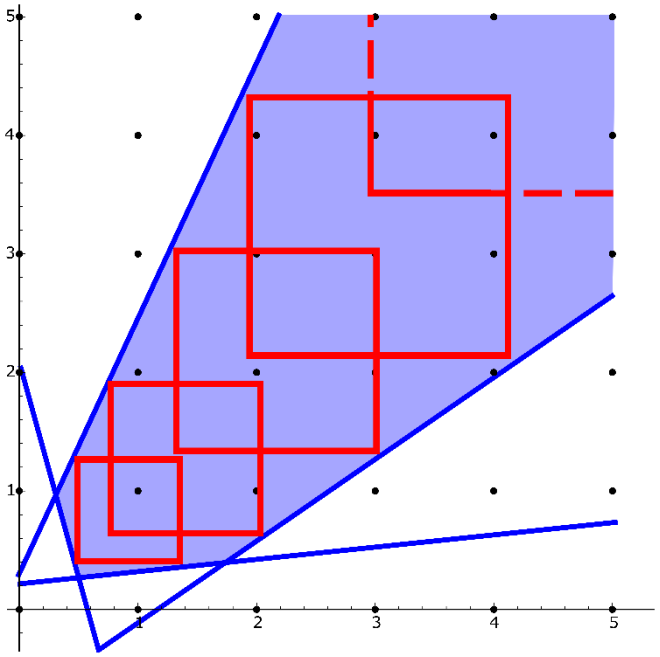
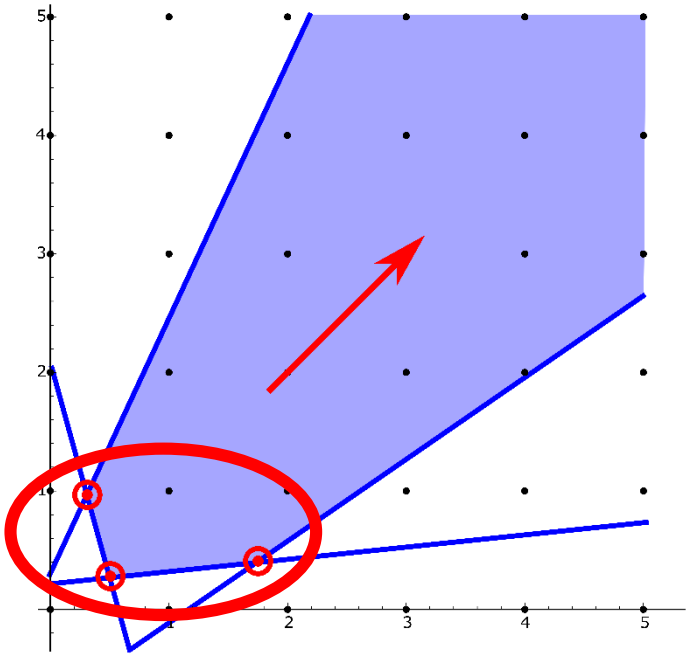
Unbounded Problems



infinite width

easy to solve with cubes

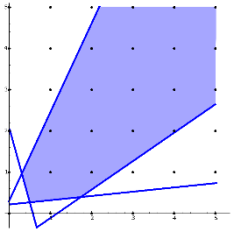
Unbounded Problems



infinite width

bad for branch-and-bound based on simplex

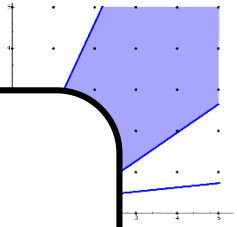
SMT-LIB Benchmarks



- 980 infinite width problems
- timeout after 10 min (per problem)
- all solvers use branch-and-bound based on simplex

Solver:	#Solved	#Unsolved	Total Time /wo Timeouts
cvc4	843	147	21113 sec
yices-2	851	129	19791 sec
z3	856	124	1672 sec
mathsat5	928	52	13304 sec
SPASS-IQ	948	32	1026 sec
SPASS-IQ & cubes	980	0	57 sec

Dillig Benchmarks



with unit cube test:

- test solves all problems
- speed-up: several orders of magnitude

Solver:	#Solved	#Unsolved	Total Time /wo Timeouts
cvc4	843	147	21113 sec
yices-2	851	129	19791 sec
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mathsat5	928	52	13304 sec
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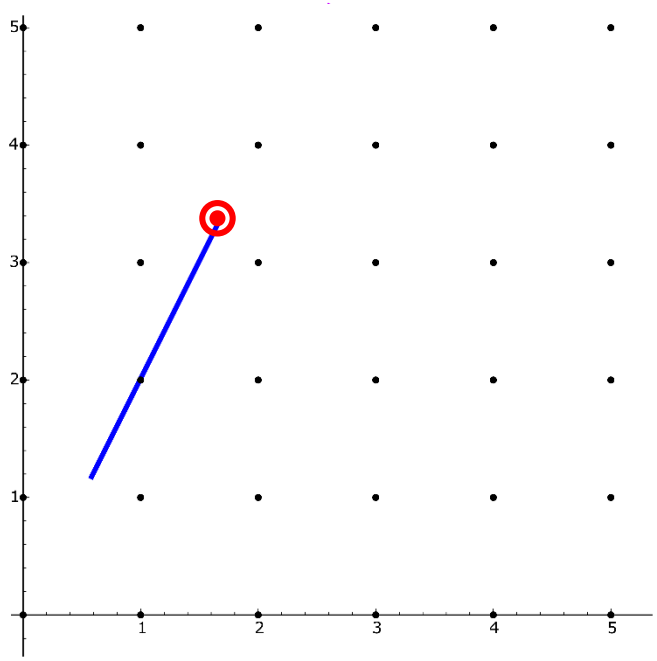
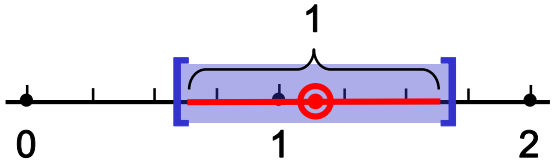
Equalities





$x \mapsto 1$ ⇨ $y \mapsto 2x$ ⇨ $y \mapsto 2$

$$2y \leq 5x, \qquad 3y \geq 4x,$$

$$2y \leq -5x + 15, \qquad 2y \geq -3x + 4,$$

$$y = 2x$$

Diophantine Equation Handler

A Practical Approach to

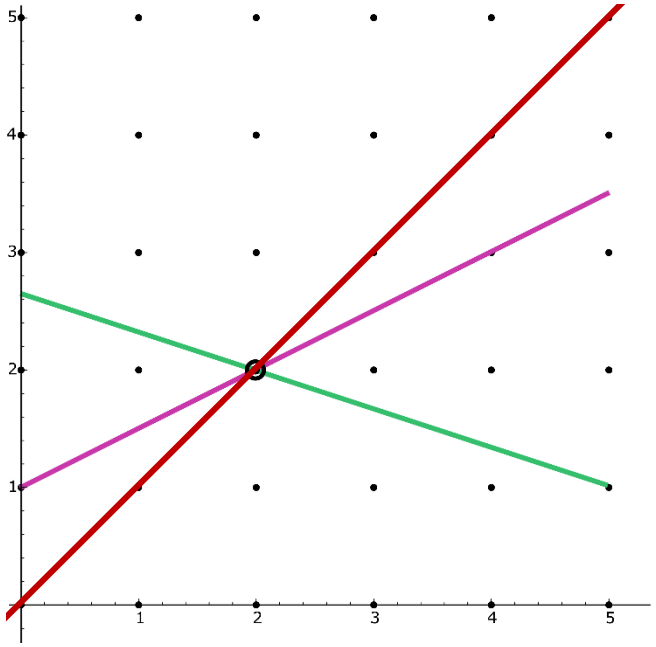
Satisfiability Modulo Linear Integer Arithmetic

by A. Griggio. JSAT 2012

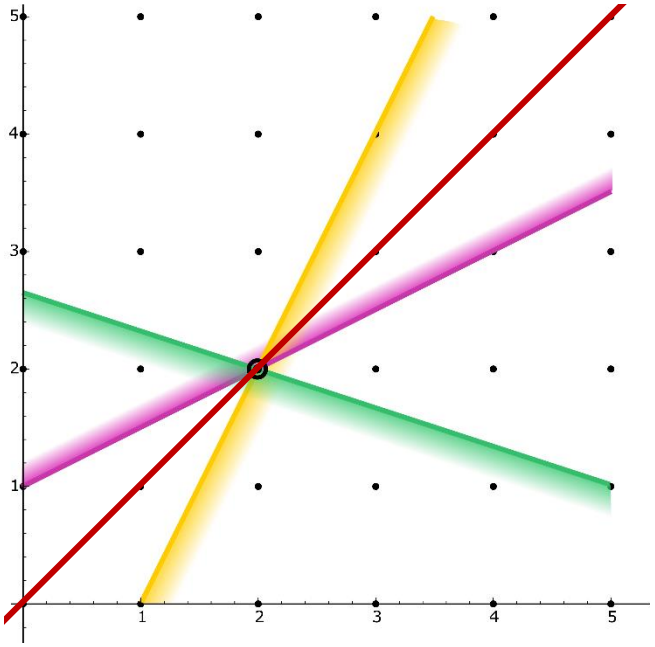
$$0 \leq x, \qquad x \geq 0,$$

$$3x \leq 5, \qquad 7x \geq 4,$$

Implied Equalities



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



explicit

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

implicit

$$2y = x + 2,$$

$$3y \leq -x + 8, \qquad 3y \geq -x + 8,$$

$$2y = x + 2$$

$$3y = -x + 8$$

$$y = 2x - 2,$$

$$2y \geq x + 2,$$

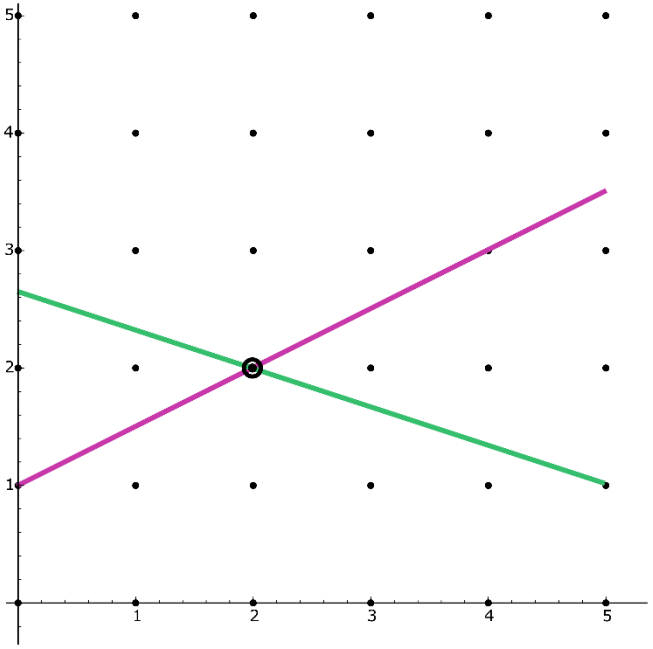
$$3y \leq -x + 8, \qquad y \leq 2x - 2,$$

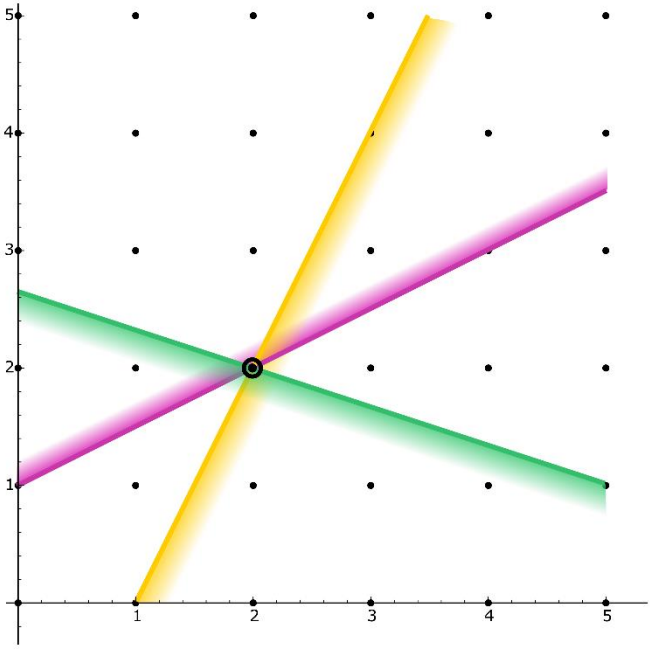
$$2y = x + 2$$

$$3y = -x + 8$$

$$y = 2x - 2,$$

Equality Basis





explicit

$$\forall x \in \mathbb{R}^n . Ax \leq b \rightarrow d^T x = c$$

implicit

$$2y = x + 2,$$
$$3y \leq -x + 8, \qquad 3y \geq -x + 8,$$

$$2y = x + 2$$
$$3y = -x + 8$$
$$y = 2x - 2,$$

$$2y \geq x + 2,$$
$$3y \leq -x + 8, \qquad y \leq 2x - 2,$$

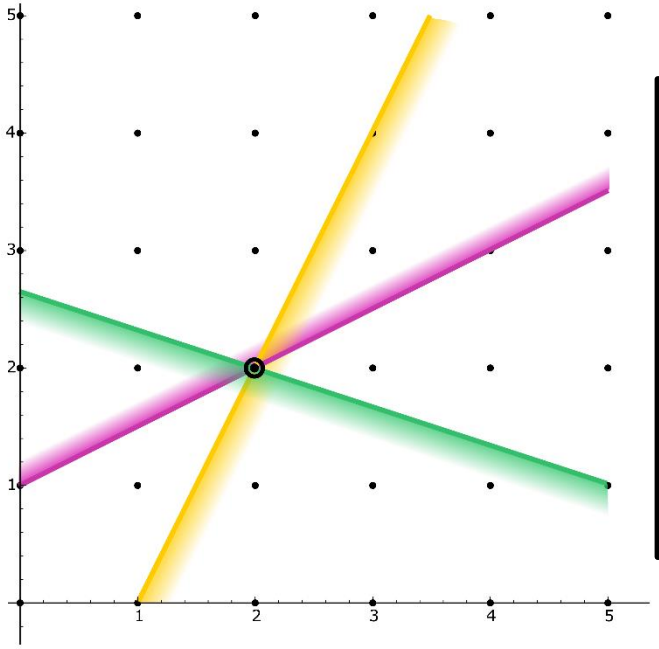
$$2y = x + 2$$
$$3y = -x + 8$$
$$y = 2x - 2,$$

Equality Basis

Equality Basis:

1.set of linear independent equalities

2. maximal



Implied Equalites:

All linear combinations of the Equality Basis

$$\forall x \in \mathbb{R}^n . Ax \leq b \rightarrow d^T x = c$$

$$2y \geq x + 2,$$
$$3y \leq -x + 8, \qquad y \leq 2x - 2,$$

$$2y = x + 2$$
$$3y = -x + 8 \qquad y = 2x - 2,$$

Simplification:

Eliminating Equalities via Substitution

Largest Cube Test

subject to:

$$a_i^T x \leq b_i \quad \text{for } i = 1, \ldots, m$$

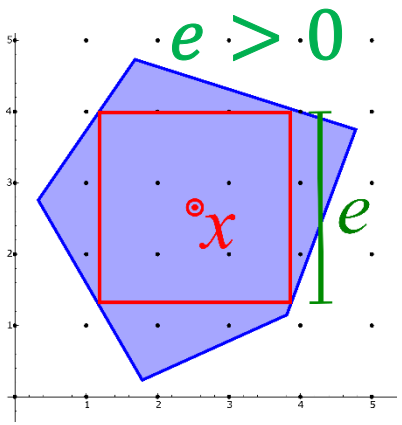
$\Rightarrow$

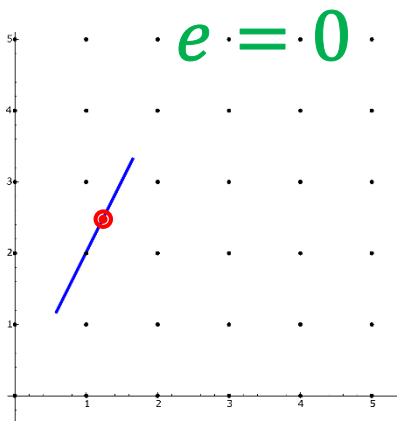
maximize e

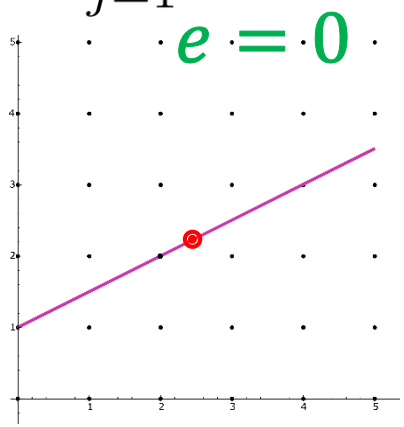
subject to:

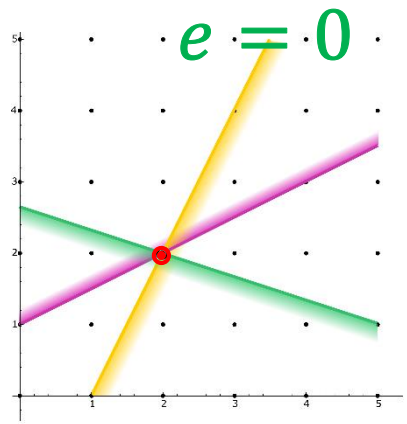
$$a_i^T x + \|a_i\|_1 \frac{e}{2} \leq b_i \quad \text{for } i = 1, \ldots, m$$

where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$









Equality Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

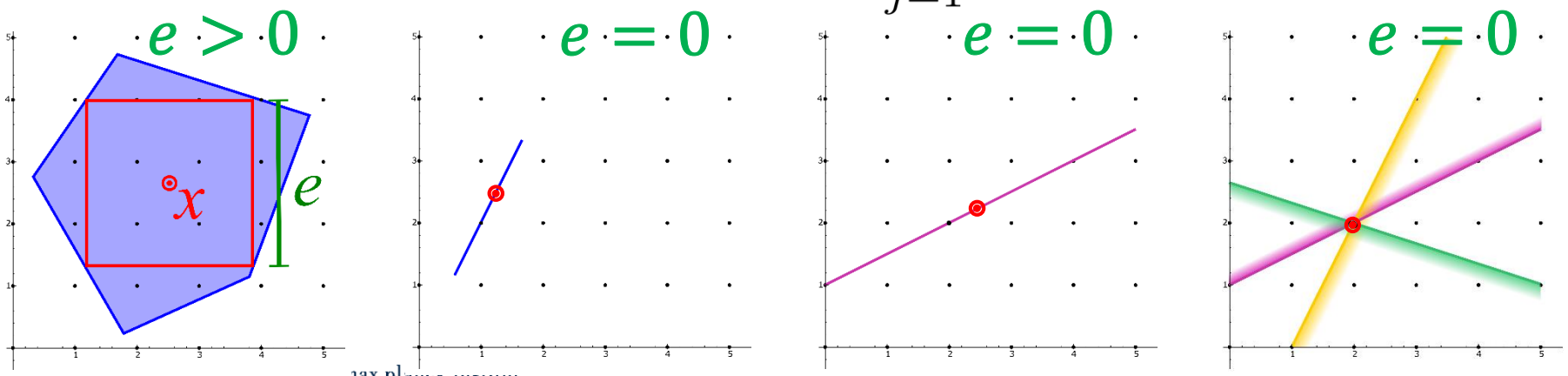
$e > 0$

subject to:

$$a_i^T x + e \leq b_i$$

for $i = 1, \ldots, m$

where $\|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$

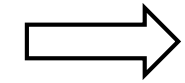


Equality Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

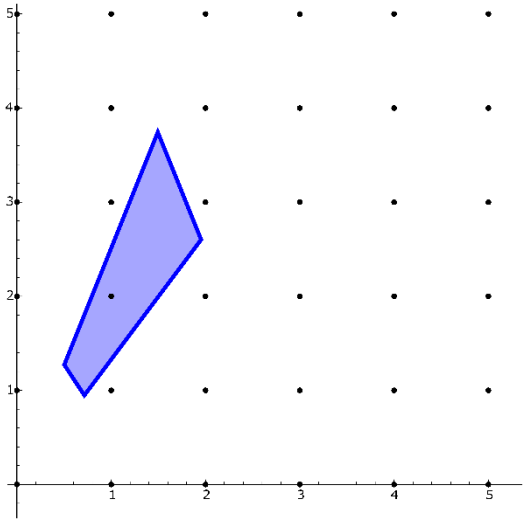


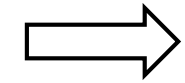
subject to:

$$a_i^T x < b_i$$

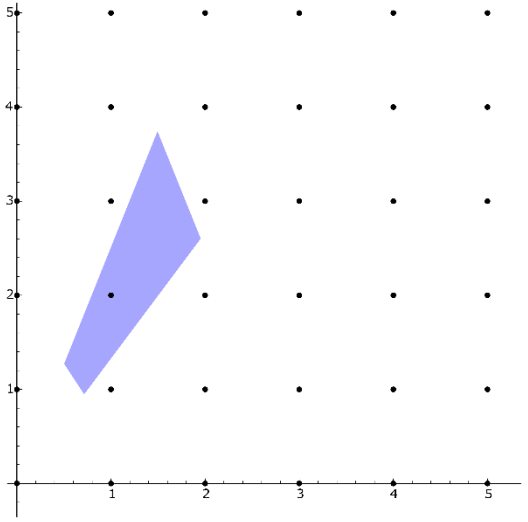
for $i = 1, \ldots, m$

Interior & Surface





Interior

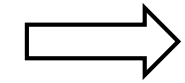


Equality Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

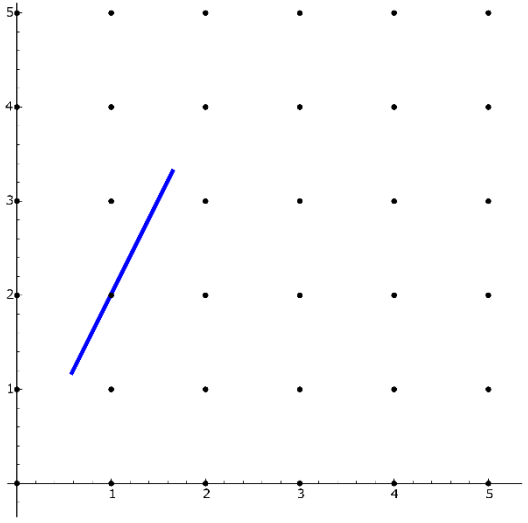


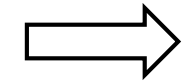
subject to:

$$a_i^T x < b_i$$

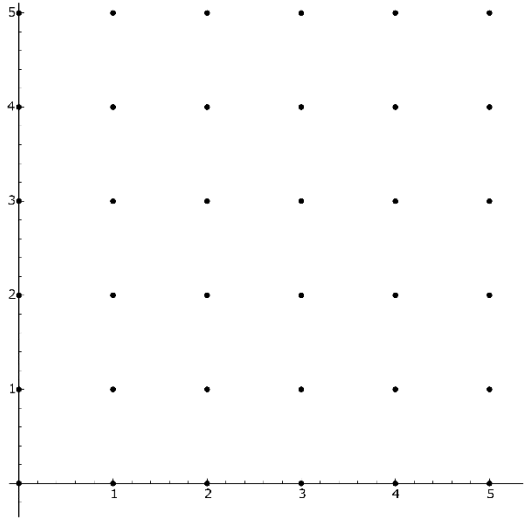
for $i = 1, \ldots, m$

Interior & Surface





Interior

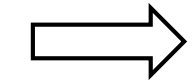


Equality Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

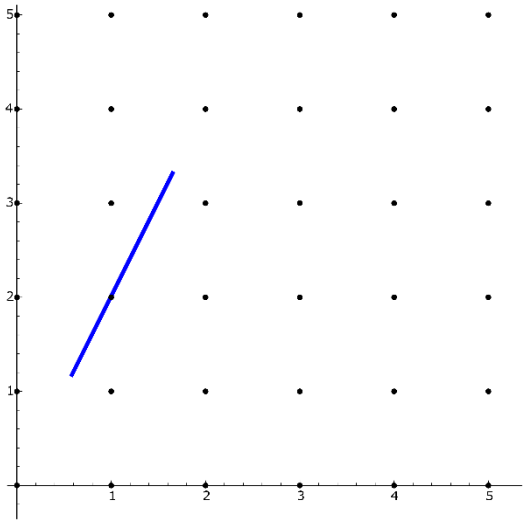


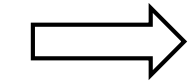
subject to:

$$a_i^T x < b_i$$

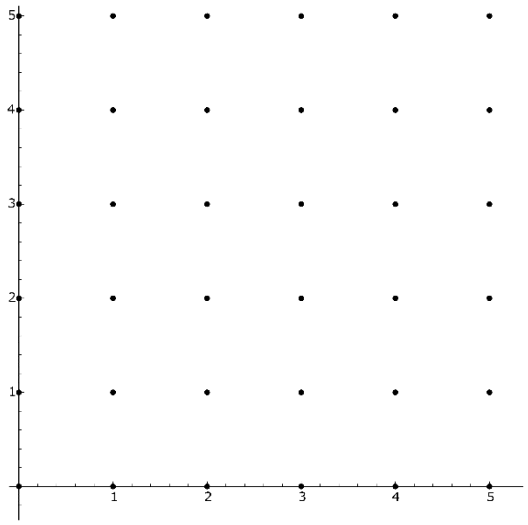
for $i = 1, \ldots, m$

sat





unsat

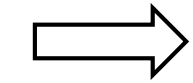


Equality Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

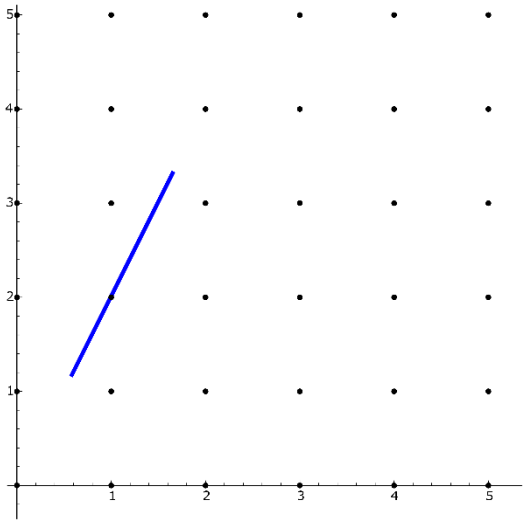


subject to:

$$a_i^T x < b_i$$

for $i = 1, \ldots, m$

sat



unsat

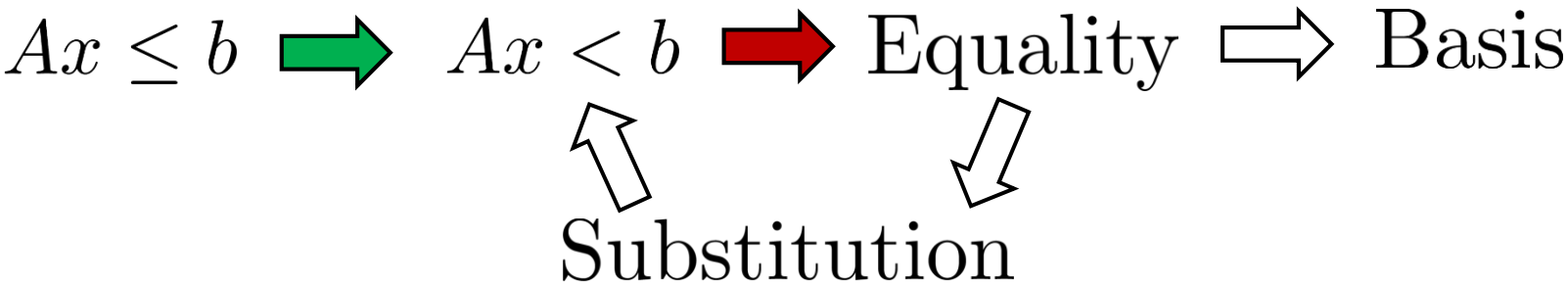
Conflict: $C \subseteq \{1, \ldots, m\}$

$$a_i^T x < b_i \quad \text{for } i \in C$$

unsat

$$a_i^T x = b_i \quad \text{for } i \in C$$

Computing an Equality Basis



Computing a Complete Basis for Equalities Implied by a System of LRA Constraints

Bromberger & Weidenbach, SMT Workshop 2016

Talk: Saturday, July 2nd, 11:00

Conclusions

Tests based on cubes are:

- an intuitive concept
- easy to implement
- useful in practice
- easy to extend: equalities

Thank you for your attention!