Intuitionistic Layered Graph Logic

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Joint work with David Pym

Outline

Complex Systems and Layering

Intutionistic Layered Graph Logic

Modelling

Metatheory

Complex Systems and Layering

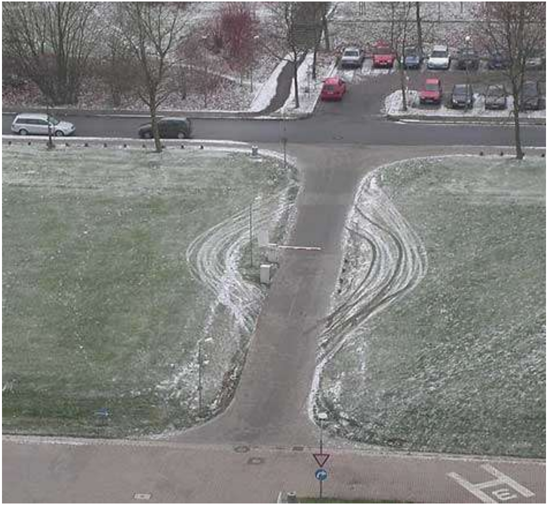
Layering In The Wild

- A complex system can be thought of a structure comprised of interconnected and interacting layers.
- More broadly: The IP stack, access control models, distributed systems, bus networks[1].
- Issues in security often arise due to a mismatch between policy and the structure of the layers of the system it applies to[2].

[1] M. Kurant and P. Thiran. Layered complex networks. *Phys. Rev. Lett.*. 96:138701. 2006

[2] T. Caulfield and D. Pym. Modelling and simulating system security policy. *Proceedings SIMUTools '15*, 9-18, 2015

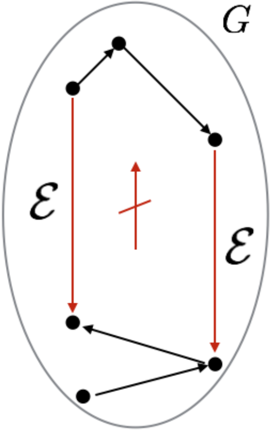
Schneier’s Gate



https://www.schneier.com/blog/archives/2005/02/the_weakest_lin.html

A Mathematical Definition Of Layering

Let $\mathcal{G}$ be an *ambient* directed graph, $\mathcal{E}$ a non-empty subset of $\mathcal{G}$'s edges and G a subgraph.



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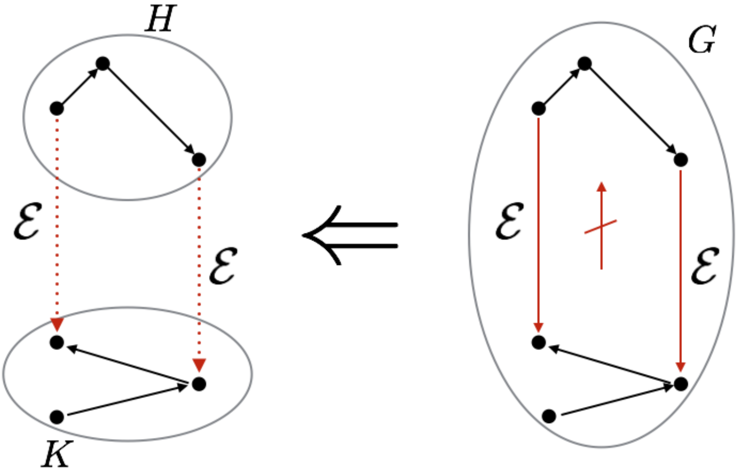
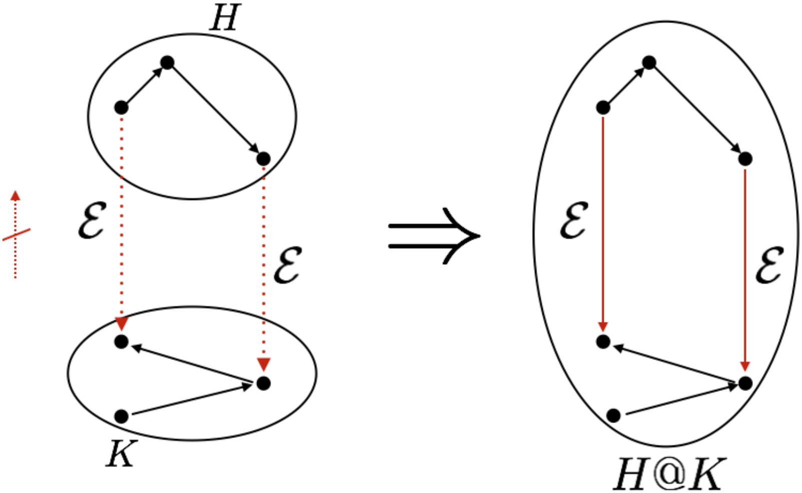


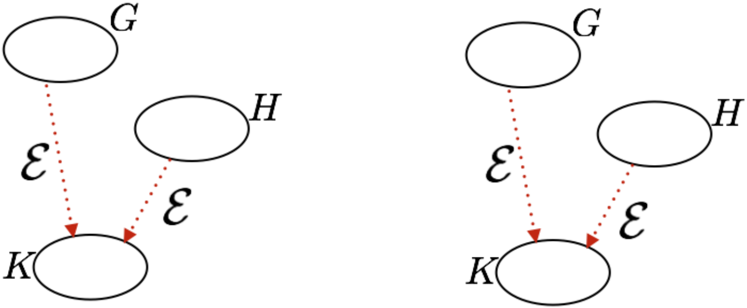
A Mathematical Definition Of Layering

This decomposition determines a *layering composition* operator @ on subgraphs of $\mathcal{G}$.



Properties of layering

The layering operation @ on subgraphs is *partial*, *non-commutative* and ***non-associative***.



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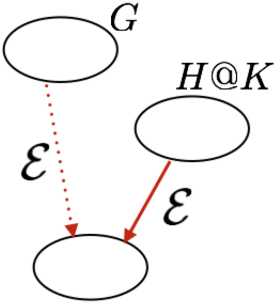

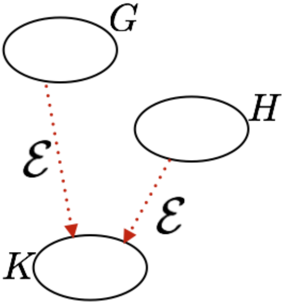

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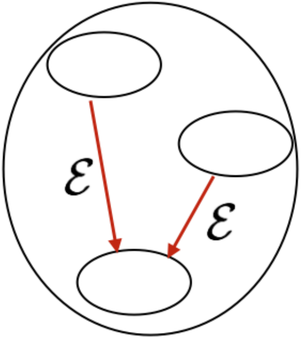
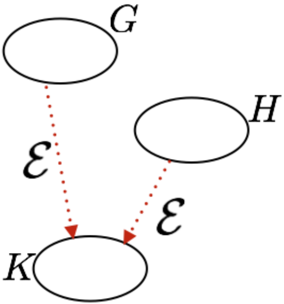


$G@(H@K)$



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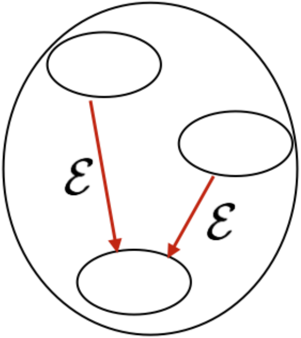

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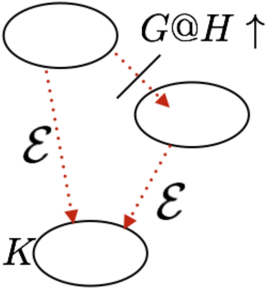

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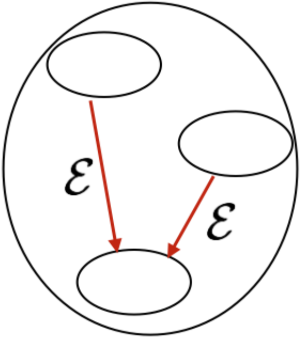
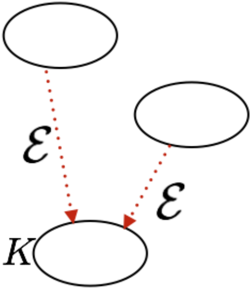


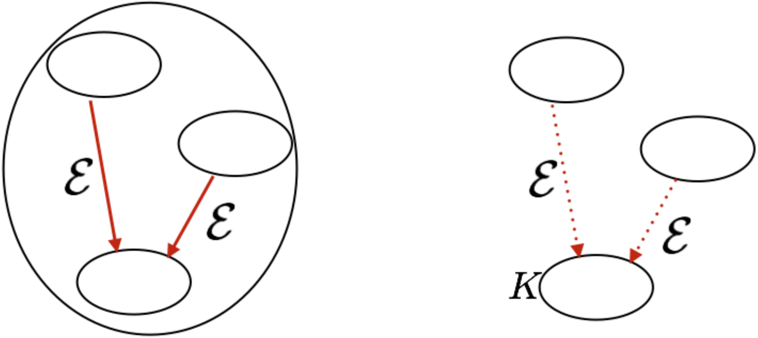
$G@(H@K)$



$(G@H)@K\uparrow$

Properties of layering

The layering operation @ on subgraphs is *partial*, *non-commutative* and ***non-associative***.



$$G@(H@K) \neq (G@H)@K$$

Layered Graph Logic

- LGL, a substructural logic for reasoning about graph layering, has been given[3] and developed into an access control assertion language[4].
- LGL lacks desirable metatheoretic properties for its layered graph semantics.
- Intuitionistic variant ILGL overcomes these deficiencies.

[3] M. Collinson, K. McDonald, and D. Pym. A substructural logic for layered graphs. *Journal of Logic and Computation*, 24(4):953–988, 2014

[4] M. Collinson, K. McDonald, and D. Pym. Layered graph logic as an assertion language for access control policy models. *Journal of Logic and Computation*, 2015. doi=10.1093/logcom/exv020.

Intutionistic Layered Graph Logic

Syntax

$$\phi ::= \mathrm{p} \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \phi \blacktriangleright \phi \mid \phi {-\!\!\blacktriangleright} \phi \mid \phi {\blacktriangleright\!\!-} \phi$$

- Additive fragment: intuitionistic propositional logic.
- Multiplicative fragment: non-associative Lambek calculus

Syntax

$$\phi ::= \mathrm{p} \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \blacktriangleright \phi \mid \phi \mathrel{-\!\!\blacktriangleright} \phi \mid \phi \mathrel{\blacktriangleright\!\!-} \phi$$

- Additive fragment: intuitionistic propositional logic.
- Multiplicative fragment: non-associative Lambek calculus

$$\frac{}{\varphi \vdash \varphi}\,(\mathrm{Ax}) \qquad \frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \chi}\,(\mathrm{Cut}) \qquad \frac{}{\varphi \vdash \top}\,(\top) \qquad \frac{}{\bot \vdash \varphi}\,(\bot)$$

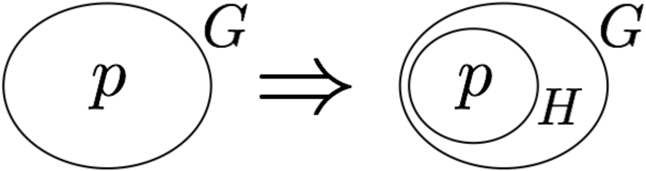
$$\frac{\varphi \vdash \psi \qquad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}\,(\wedge_1) \qquad \frac{}{\varphi_1 \wedge \varphi_2 \vdash \varphi_i}\,(\wedge_2) \qquad \frac{}{\varphi_i \vdash \varphi_1 \vee \varphi_2}\,(\vee_1) \qquad \frac{\varphi \vdash \chi \qquad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}\,(\vee_2)$$

$$\frac{\varphi \vdash \psi \to \chi \qquad \nu \vdash \psi}{\varphi \wedge \nu \vdash \chi}\,(\to_1) \qquad \frac{\varphi \wedge \psi \vdash \chi}{\varphi \vdash \psi \to \chi}\,(\to_2) \qquad \frac{\varphi \vdash \psi \qquad \chi \vdash \upsilon}{\varphi \blacktriangleright \chi \vdash \psi \blacktriangleright \upsilon}\,(\blacktriangleright)$$

$$\frac{\varphi \vdash \psi \mathrel{-\!\!\blacktriangleright} \chi \qquad \upsilon \vdash \psi}{\varphi \blacktriangleright \upsilon \vdash \chi}\,(\mathrel{-\!\!\blacktriangleright}_1) \qquad \frac{\varphi \blacktriangleright \psi \vdash \chi}{\varphi \vdash \psi \mathrel{-\!\!\blacktriangleright} \chi}\,(\mathrel{-\!\!\blacktriangleright}_2) \qquad \frac{\varphi \vdash \psi \mathrel{\blacktriangleright\!\!-} \chi \qquad \upsilon \vdash \psi}{\upsilon \blacktriangleright \varphi \vdash \chi}\,(\mathrel{\blacktriangleright\!\!-}_1) \qquad \frac{\varphi \blacktriangleright \psi \vdash \chi}{\psi \vdash \varphi \mathrel{\blacktriangleright\!\!-} \chi}\,(\mathrel{\blacktriangleright\!\!-}_2)$$

Semantics

Let $\mathcal{G}$ be a graph, $\mathcal{E}$ a set of its edges, X a set of its subgraphs closed under @ and $\mathcal{V} : \mathrm{Prop} \to P(X)$ a valuation satisfying *persistence*: if $G \in \mathcal{V}(p)$ and $H \sqsubseteq G$ then $H \in \mathcal{V}(p)$.

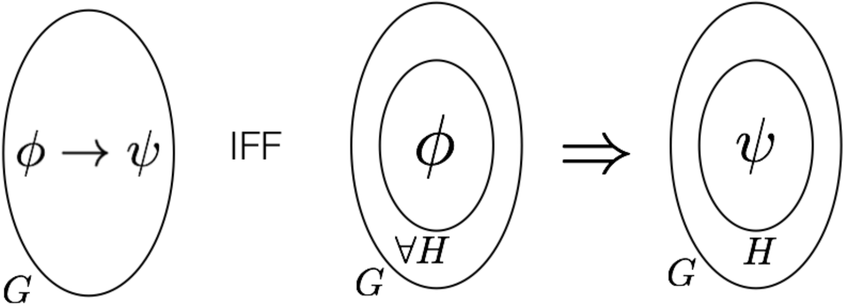


$G \vDash \top$ always $\quad G \vDash \bot$ never $\quad G \vDash \mathrm{p}$ iff $G \in \mathcal{V}(\mathrm{p})$

$G \vDash \varphi \wedge \psi$ iff $G \vDash \varphi$ and $G \vDash \psi \quad G \vDash \varphi \vee \psi$ iff $G \vDash \varphi$ or $G \vDash \psi$

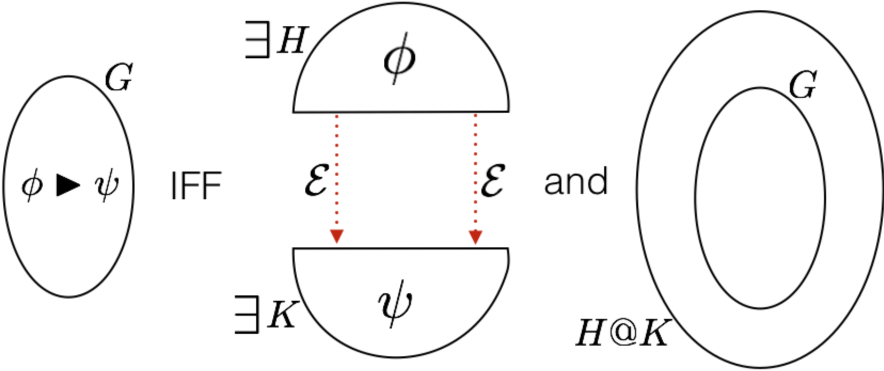
Additive Implication

$$G \vDash \phi \rightarrow \psi \text{ iff } \forall H \sqsubseteq G : \text{ if } H \vDash \phi \text{ then } H \vDash \psi$$



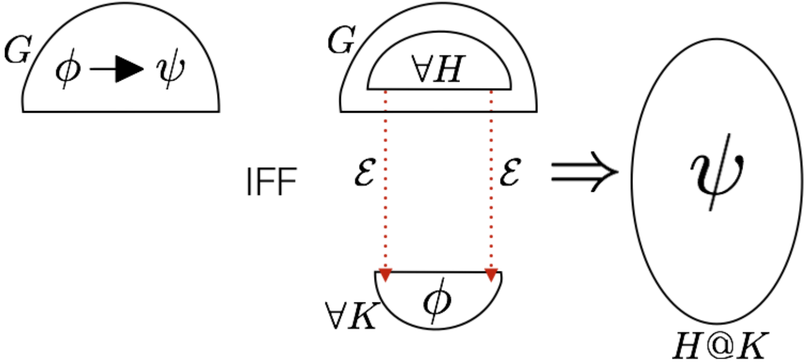
Multiplicative Conjunction

$$G \vDash \phi \blacktriangleright \psi \text{ iff } \exists H, K : H@K\downarrow, H \vDash \phi, K \vDash \psi \text{ and } G \sqsubseteq H@K$$



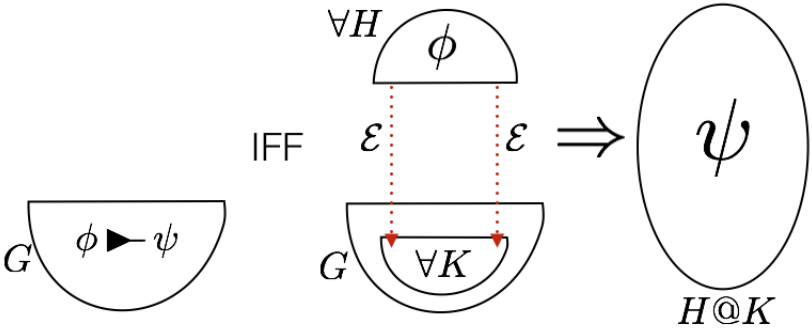
Multiplicative Implication 1

$G \vDash \phi \mathbin{-\!\!\blacktriangleright} \psi$ iff $\forall H, K :$ if $H \sqsubseteq G$, $H@K \downarrow$ and $K \vDash \phi$ then $H@K \vDash \psi$



Multiplicative Implication 2

$G \vDash \phi \blacktriangleright\!\!\!- \psi$ iff $\forall H, K$: if $K \sqsubseteq G$, $H@K \downarrow$ and $H \vDash \phi$ then $H@K \vDash \psi$



Modelling

Bunched Logic

- ILGL is an instance of a *bunched logic*[5].
- The bunched logics BI and BBI underpin *separation logic*[6] used in program verification.
- Frame rule + bi-abduction[7] = industrial applications (Facebook)
- LGL (ILGL) + commutativity + associativity + unit = BBI (BI).

[5] P O'Hearn, D Pym. The logic of bunched implications. *Bulletin Of Symbollic Logic* 5(2) 215-244, 1999

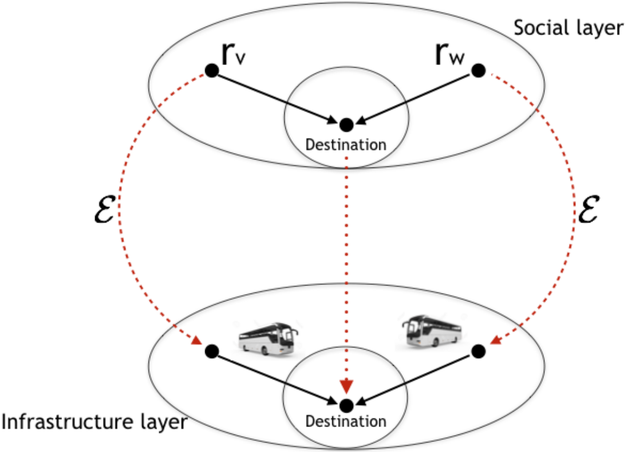
[6] J C Reynolds. Separation logic: a logic for shared mutable data structures. *Proceedings of LICS '02*, 55-74, 2002,

[7] C Calcagno et al. Compositional shape analysis by means of bi-abduction. *Proceedings POPL '09*, 289-300. 2009

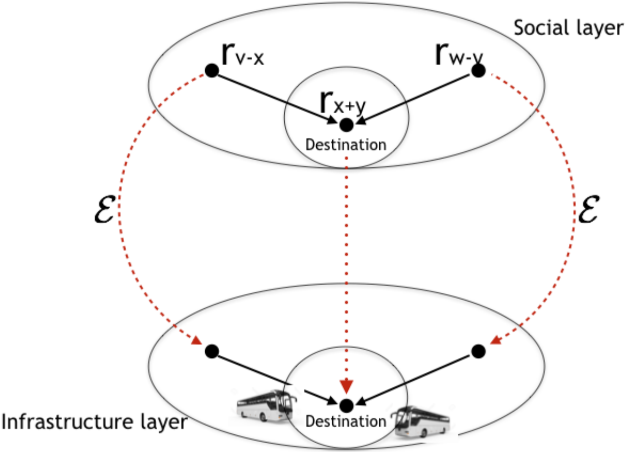
Resource Labelled Extension

- Idea: extend in the style of separation logic in order to model complex systems.
- Simple extension:
  - Vertices labelled with resources
  - Actions connected to modallties relabel vertices
  - Propositional language rich enough to express basic facts about labelling.
- Example doesn't utilise non-associativity/non-commutativity in an essential way, but gives indication of how modelling may work.

Bus Network


Social layer
r_v
r_w
Destination
$\mathcal{E}$
$\mathcal{E}$
Destination
Infrastructure layer

Bus Network



AFTER $\langle bus^a_x \rangle \langle bus^b_x \rangle$

Bus Network

- Let ϕ_x denote that buses pick up x people at the bus stops.
- Let ϕ_{meeting} denote that there is a meeting at the destination.
- Let ϕ_{quorum} denote that at least 50 people attend the meeting.
- We have $G_2 \vDash \langle bus^a_{25} \rangle \langle bus^b_{35} \rangle ((\phi_{\mathrm{meeting}} \blacktriangleright \phi_{60}) \mathrel{-\!\!\blacktriangleright} \phi_{\mathrm{quorum}})$ denoting that buses of joint capacity of 60 are sufficient to make the meeting quorate.
- We have $G_2 \vDash \langle bus^b_{40} \rangle ((\phi_{\mathrm{meeting}} \blacktriangleright \phi_{40}) \mathrel{-\!\!\blacktriangleright} \neg \phi_{\mathrm{quorum}})$ denoting that a single bus of capacity 40 is not sufficient.

Metatheory

Labelled Tableaux

$$\frac{\mathbb{T}\varphi \land \psi : x \in \mathcal{F}}{\langle\{\mathbb{T}\varphi : x, \mathbb{T}\psi : x\}, \emptyset\rangle}\langle\mathbb{T}\land\rangle \qquad \frac{\mathbb{F}\varphi \land \psi : x \in \mathcal{F}}{\langle\{\mathbb{F}\varphi : x\}, \emptyset\rangle \mid \langle\{\mathbb{F}\psi : x\}, \emptyset\rangle}\langle\mathbb{F}\land\rangle$$

$$\frac{\mathbb{T}\varphi \lor \psi : x \in \mathcal{F}}{\langle\{\mathbb{T}\varphi : x\}, \emptyset\rangle \mid \langle\{\mathbb{T}\psi : x\}, \emptyset\rangle}\langle\mathbb{T}\lor\rangle \qquad \frac{\mathbb{F}\varphi \lor \psi : x \in \mathcal{F}}{\langle\{\mathbb{F}\varphi : x, \mathbb{F}\psi : x\}, \emptyset\rangle}\langle\mathbb{F}\lor\rangle$$

$$\frac{\mathbb{T}\varphi \to \psi : x \in \mathcal{F} \text{ and } x \preccurlyeq y \in \overline{C}}{\langle\{\mathbb{F}\varphi : y\}, \emptyset\rangle \mid \langle\{\mathbb{T}\psi : y\}, \emptyset\rangle}\langle\mathbb{T}\to\rangle \qquad \frac{\mathbb{F}\varphi \to \psi : x \in \mathcal{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_i\}, \{x \preccurlyeq c_i\}\rangle}\langle\mathbb{F}\to\rangle$$

$$\frac{\mathbb{T}\varphi \blacktriangleright \psi : x \in \mathcal{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{T}\psi : c_j\}, \{c_i c_j \preccurlyeq x\}\rangle}\langle\mathbb{T}\blacktriangleright\rangle \qquad \frac{\mathbb{F}\varphi \blacktriangleright \psi : x \in \mathcal{F} \text{ and } yz \preccurlyeq x \in \overline{C}}{\langle\{\mathbb{F}\varphi : y\}, \emptyset\rangle \mid \langle\{\mathbb{F}\psi : z\}, \emptyset\rangle}\langle\mathbb{F}\blacktriangleright\rangle$$

$$\frac{\mathbb{T}\varphi \mathrel{-\!\!\blacktriangleright} \psi : x \in \mathcal{F} \text{ and } x \preccurlyeq y, yz \preccurlyeq yz \in \overline{C}}{\langle\{\mathbb{F}\varphi : z\}, \emptyset\rangle \mid \langle\{\mathbb{T}\psi : yz\}, \emptyset\rangle}\langle\mathbb{T}\mathrel{-\!\!\blacktriangleright}\rangle \qquad \frac{\mathbb{F}\varphi \mathrel{-\!\!\blacktriangleright} \psi : x \in \mathcal{F}}{\langle\{\mathbb{T}\varphi : c_j, \mathbb{F}\psi : c_i c_j\}, \{x \preccurlyeq c_i, c_i c_j \preccurlyeq c_i c_j\}\rangle}\langle\mathbb{F}\mathrel{-\!\!\blacktriangleright}\rangle$$

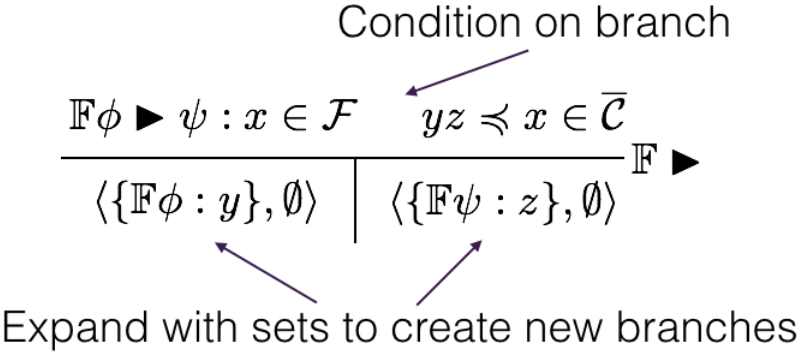
$$\frac{\mathbb{T}\varphi \mathrel{\blacktriangleright\!\!-} \psi : x \in \mathcal{F} \text{ and } x \preccurlyeq y, zy \preccurlyeq zy \in \overline{C}}{\langle\{\mathbb{F}\varphi : z\}, \emptyset\rangle \mid \langle\{\mathbb{T}\psi : zy\}, \emptyset\rangle}\langle\mathbb{T}\mathrel{\blacktriangleright\!\!-}\rangle \qquad \frac{\mathbb{F}\varphi \mathrel{\blacktriangleright\!\!-} \psi : x \in \mathcal{F}}{\langle\{\mathbb{T}\varphi : c_j, \mathbb{F}\psi : c_j c_i\}, \{x \preccurlyeq c_i, c_j c_i \preccurlyeq c_j c_i\}\rangle}\langle\mathbb{F}\mathrel{\blacktriangleright\!\!-}\rangle$$

with c_i and c_j being fresh atomic labels [8]

[8] D. Larchey-Wendling. The formal proof of the strong completeness of partial monoidal Boolean BI. *Journal of Logic and Computation*. 2014. doi:10.1093/logcom/exu031

Labelled Tableaux

A branch is a set of labelled formulae $\mathcal{F}$ and a set of inequalities on labels $\mathcal{C}$.



Condition on branch

$$\frac{\mathbb{F}\phi \blacktriangleright \psi : x \in \mathcal{F} \qquad yz \preccurlyeq x \in \overline{\mathcal{C}}}{\langle\{\mathbb{F}\phi : y\}, \emptyset\rangle \;\Big|\; \langle\{\mathbb{F}\psi : z\}, \emptyset\rangle}\ \mathbb{F}\blacktriangleright$$

Expand with sets to create new branches

A branch $\langle\mathcal{F}, \mathcal{C}\rangle$ is closed iff there exists x, y, ϕ such that either i) $\mathbb{F}\top : x \in \mathcal{B}$ or ii) $\mathbb{T}\bot : x \in \mathcal{B}$ or iii) $\mathbb{T}\phi : x, \mathbb{F}\phi : y \in \mathcal{F}$ and $x \preccurlyeq y \in \overline{\mathcal{C}}$.

A tableaux proof

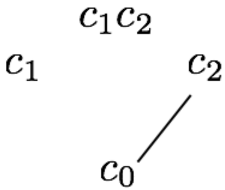
$$\mathbb{F} p \mathrel{-\!\!\blacktriangleright} (\top \blacktriangleright p) : c_0$$

A tableaux proof

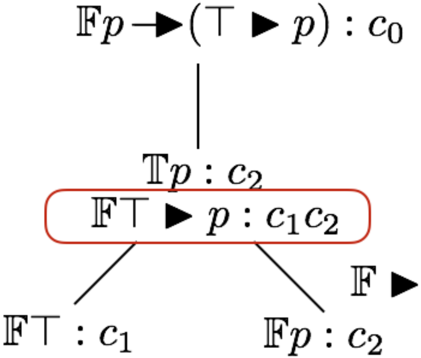
$$\boxed{\mathbb{F} p \mathrel{-\!\!\blacktriangleright} (\top \blacktriangleright p) : c_0}$$

$$\Big| \; \mathbb{F} \mathrel{-\!\!\blacktriangleright}$$

$$\mathbb{T} p : c_2$$
$$\mathbb{F} \top \blacktriangleright p : c_1 c_2$$



A tableaux proof



$\mathbb{F} p \mathrel{-\!\!\blacktriangleright} (\top \blacktriangleright p) : c_0$

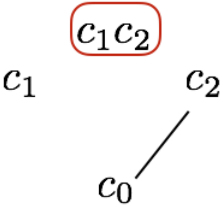
$\mathbb{T} p : c_2$

$\boxed{\mathbb{F} \top \blacktriangleright p : c_1 c_2}$

$\mathbb{F} \blacktriangleright$

$\mathbb{F} \top : c_1$

$\mathbb{F} p : c_2$



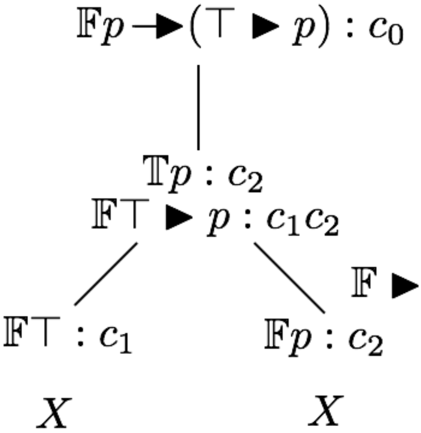
$\boxed{c_1 c_2}$

c_1

c_2

c_0

A tableaux proof



$\mathbb{F} p \mathrel{-\!\!\blacktriangleright} (\top \blacktriangleright p) : c_0$

$\mathbb{T} p : c_2$

$\mathbb{F} \top \blacktriangleright p : c_1 c_2$

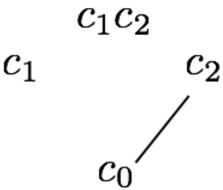
$\mathbb{F} \blacktriangleright$

$\mathbb{F} \top : c_1$

$\mathbb{F} p : c_2$

X

X



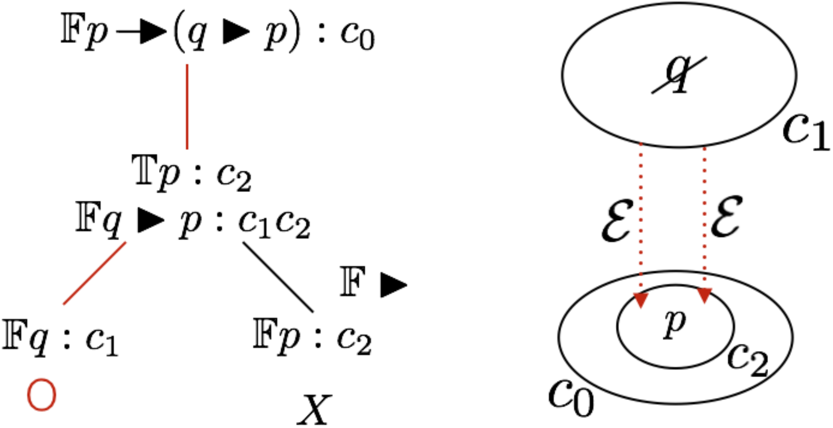
$c_1 c_2$

c_1

c_2

c_0

Countermodel construction


$\mathbb{F}p \multimap\!\blacktriangleright (q \blacktriangleright p) : c_0$
$\mathbb{T}p : c_2$
$\mathbb{F}q \blacktriangleright p : c_1c_2$
$\mathbb{F}\blacktriangleright$
$\mathbb{F}q : c_1$
O
$\mathbb{F}p : c_2$
X
$\cancel{q}$
c_1
$\mathcal{E}$
$\mathcal{E}$
p
c_2
c_0

Soundness And Completeness

Theorem

ϕ is valid in the graph theoretic semantics iff there exists a closed tableau for ϕ.

Alternative Semantics

Algebraic Semantics: A layered Heyting algebra is a structure $(\mathcal{A}, \wedge, \vee, \to, \bot, \top, \blacktriangleright, -\!\!\blacktriangleright, \blacktriangleright\!\!-)$ such that $(\mathcal{A}, \wedge, \vee, \to, \bot, \top)$ is a Heyting algebra and $(A, \leq, \blacktriangleright, -\!\!\blacktriangleright, \blacktriangleright\!\!-)$ is a residuated groupoid:

$$a \blacktriangleright b \leq c \text{ iff } a \leq b -\!\!\blacktriangleright c \text{ iff } b \leq a \blacktriangleright\!\!- c$$

Theorem

ϕ is valid on layered Heyting algebras iff $\vdash \phi$.

Relational Semantics A relational frame is a structure $(X, \preccurlyeq, R)$ such that $\preccurlyeq$ is a preorder and $R \subseteq X^3$.

Theorem

ϕ is valid on relational frames iff a closed tableau for ϕ exists.

Equivalences

Theorem (Representation Theorem)

1. *Every relational frame generates a layered Heyting algebra.*
2. *Every layered Heyting algebra can be embedded in a concrete layered Heyting algebra generated by a relational frame.*

Corollary

$\vdash \phi$ *iff* ϕ *valid on algebras iff* ϕ *valid on graphs iff there exists a closed tableau for* ϕ*.*

Theorem

The category of layered Heyting algebras is dually equivalent to the category of ILGL spaces.

Decidability

A variety of algebras has the *finite embeddability property* (FEP) iff for any algebra $\mathcal{A}$ and finite subset $\mathcal{B}_0 \subseteq \mathcal{A}$, there exists a finite algebra $\mathcal{B}$ and a homomorphic embedding $\mathcal{B}_0 \to \mathcal{B}$.

Theorem

The variety of layered Heyting algebras has the FEP.

Corollary

ILGL has the finite model property.

Proof.

$\mathcal{A}$ is (possibly infinite) countermodel for invalid ϕ.
$\mathcal{B}_0 = \{[\![\psi]\!] \mid \psi$ subformula of $\phi\}$. $\mathcal{B}$ is a finite countermodel for ϕ.

□

Future Work

- Modal and/or separation logic style extensions for modelling.
- Tool development: simulation modelling[9] and theorem proving (via tableaux[10]).
- Connections to intuitionistic modal logic (on a ternery relation).
- Algebraic/topological techniques for bunched logics/separation logic.

[9] M. Collinson, B. Monahan and D. Pym. A discipline of mathematical systems modelling. *College Publications*. 2012

[10] F. Béal, D Méry and D. Galmiche. B I L L: A theorem prover for propositional BI logic. `webloria.loria.fr/~dmery/tools/BILL`

Conclusions

- Well motivated substructural logic with Kripke semantics on graphs, sound and complete for labelled tableaux and Hilbert-style proof systems.
- Potential for complex system modelling.
- Equivalence of proof systems via equivalence of algebraic and relational semantics.
- Countermodel extraction that produces graph models.
- Decidability via finite model property.
- Case study for algebraic/topological methods in bunched/seperation logic.