Inducing syntactic cut-elimination for indexed nested sequents

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Preliminaries. Modal logics. Proof calculi.

- Let **Cp** denote classical propositional logic. Finitely axiomatisable with *modus ponens*. Choose your favourite connectives. Normal modal logic **K** obtained by the addition of the normality axiom and *necessitation* rule.

$$\frac{A \to B \quad B}{B}\ mp \qquad \Box(p \to q) \to (\Box p \to \Box q) \qquad \frac{A}{\Box A}\ (\Box)$$

- (Gentzen 1934) introduced the sequent calculus for classical logic. Subsequently extended to many other logics including **K**.

- The sequent calculus is a proof calculus where the basic objects are *sequents* $X \vdash Y$ where X and Y are lists/multisets of formulae.

- $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ interpreted as formula $A_1 \wedge \ldots \wedge A_n \to B_1 \vee \ldots \vee B_m$

- What's wrong with the previous proof calculus? It breaks the *subformula* property. The sequent calculus framework (i.e. use of sequents) permits a proof calculus with the subformula property.

The subformula property is key

- A proof calculus with the subformula property is *key* to using the proof calculus for metalogical argument (e.g. decidability, interpolation, consistency, density elimination). because we get a good handle on the formulae that can appear in the proof.

- Unfortunately for many logics, the sequent calculus framework seems not to yield a proof calculus with the subformula property.

- E.g. there are sequent calculi for modal logic $\mathbf{KT} = \mathbf{K} + \Box p \rightarrow p$ and $\mathbf{S4} = K + \Box p \rightarrow p + \Box p \rightarrow \Box\Box p$ but not when we add $\Diamond p \rightarrow \Box\Diamond p$ to **S4**!

- Solution: extend the framework just enough that you can get a proof calculus with subformula property for your logic.

- Typically, addition of new proof rules requires significant reworking of cut-elimination proof (assuming it works at all).

Nested Sequent Formalism (Kashima 1994, Brünnler 2006): Extend Gentzen framework via nesting

- Rather than a calculus where each premise/conclusion is a sequent, use a *tree* of sequents. Here $X, Y, P, Q, \ldots$ multisets.

$$X \vdash Y, [P \vdash Q, [U \vdash V], [L \vdash M]], [S \vdash T]$$
$$X \vdash Y, [U \vdash V, [L \vdash M], [S \vdash T]]$$

- Let's draw each nested sequent as a tree (just notation):

Nested sequents calculi: proof rules and intuition

▶ In the end we want to derive a sequent $X \vdash Y$. We extended the framework to a tree to obtain cut-elimination.

▶ Viewing the nested sequent as a tree, propositional connectives at a node behave just like in the sequent calculus. Notation: in $\Gamma\{\,\}$ let $\{\,\}$ denote a 'hole'.

$$\frac{\Gamma\{X \vdash Y, A, B, \Delta\}}{\Gamma\{X \vdash Y, A \vee B, \Delta\}} (\vee r) \qquad \frac{\Gamma\{X, A, B \vdash Y, \Delta\}}{\Gamma\{X, A \wedge B \vdash Y, \Delta\}} (\wedge l)$$

▶ How about the rules introducing $\Box$?

$$\frac{\Gamma\{\Box A, X \vdash Y, \Delta, [A, U \vdash V, \Sigma]\}}{\Gamma\{\Box A, X \vdash Y, \Delta, [U \vdash V, \Sigma]\}} (\Box l) \qquad \frac{\Gamma\{X \vdash Y, \Delta, [\;\vdash A]\}}{\Gamma\{X \vdash Y, \Delta, \Box A\}} (\Box r)$$

▶ How to understand the construction of these rules? To follow.

Nested sequents: some results and limitations

- Cut-free calculi for intuitionistic logic, conditional logics, logics in the classical and intuitionistic modal cube and path axiom extensions of classical modal logic.
- So we have a natural generalisation of the sequent calculus which captures lots of logics!
- But.. the proof of cut-elimination is complicated each and every time. . . (some of the above results are for classes of logics, but still. . .)
- The difficulty is that each extension of **K** is a distinct rule extension and requires a distinct proof of cut-elimination

A proof framework motivated by frame semantics

- Actually lots of different proof frameworks have been proposed e.g. labelled sequent calculi, hypersequent calculi, display calculi.
- The other relevant framework for the talk today: calculus built from *labelled sequents* $\mathcal{R}, X \vdash Y$ where $\mathcal{R}$ is a multiset of Rxy terms, X and Y are multisets of formulae $x : A$.
- Frame semantics for modal logics. Directed graphs where the edge $x \mapsto y$ is denoted Rxy. So that's where the $\mathcal{R}$ comes from.
- $A \in \mathbf{K}$ iff A is valid on all frames. Validity on a frame means evaluating to true at all worlds under all valuations (assign a subset of the propositional variables to each node in the frame).
- $\Box A$ is true at a world u of a frame F under a given valuation if A is true at every v such that Ruv. $\Diamond A$ is 'there exists a world $v \ldots$'.
- $A \in \mathbf{KT}$ iff A is valid on all reflexive frames. $A \in \mathbf{S4}$ iff A is valid on all reflexive transitive frames.

Labelled sequent calculus (Fitting 1983, Mints 1997)

- Calculus built from labelled sequents $\mathcal{R}, X \vdash Y$ where $\mathcal{R}$ is a multiset of Rxy terms, X and Y are multisets of labelled formulae $x : A$.

- Implicitly quantify over all the state variables x and y such that Rxy. Then a labelled sequent can be interpreted as a statement that holds on all frames. So we can understand $\vdash x : p \to p$ and $Rxy, Ryz, x : \Box\Box p \vdash z : p$.

- (Negri 2005) has shown that modal logics whose frames are characterised by geometric first-order formulae have a cut-free labelled sequent calculus (uniform proof).

$$\frac{Rxx, X \vdash Y}{X \vdash Y}\text{(refl)} \qquad \frac{Rxy, Ryz, Ryz, X \vdash Y}{Rxy, Ryz, X \vdash Y}\text{(trans)} \qquad \frac{Rxy, X \vdash Y, y : A}{X \vdash Y, x : \Box A}(\Box\text{r})^*$$

- *variable restriction: y does not appear in the conclusion.

- Side comment. The key to such uniform proofs of cut-elimination is the use of *structural rules*.

Relating nested sequents and labelled sequents

- (Goré and R. 2012) consider how to relate the nested sequent with the labelled sequent with the aim of porting results from former to latter.
- A labelled sequent $\mathcal{R}, X \vdash Y$ whose relation multiset *defines* a tree (draw the graph defined by $\mathcal{R}$) is a nested sequent
- Call such a labelled sequent a *labelled tree sequent* LTS
- In a concrete calculus, if we can prove cut-elimination on labelled sequents *always restricted to trees*, then this is immediately cut-elimination for the corresponding nested sequent calculus.

Relating nested sequents and labelled sequents (cont.)

- Compare the NS and LS rules ($\Box$r)

$$\frac{\Gamma\{X \vdash Y, \Delta, [\ \ \vdash A]\}}{\Gamma\{X \vdash Y, \Delta, \Box A\}}\ (\Box r) \qquad \frac{Rxy, X \vdash Y, y : A}{X \vdash Y, x : \Box A}\ (\Box r)^{*}$$

- Think of the underlying trees and see past the notation
- $[\ \ \vdash A]$ corresponds to variable restriction $*$
- This correspondence was used to answer a question about the relationship between two calculi for modal provability logic *GL*
- In general however how much we can import from LS to NS is limited due to the restriction that $\mathcal{R}$ is a tree
- In particular, we don't get a general cut-elimination theorem for NS calculi from the general result for LS calculi
- LS proof relies on more general structures than trees (substitution lemma breaks tree property)

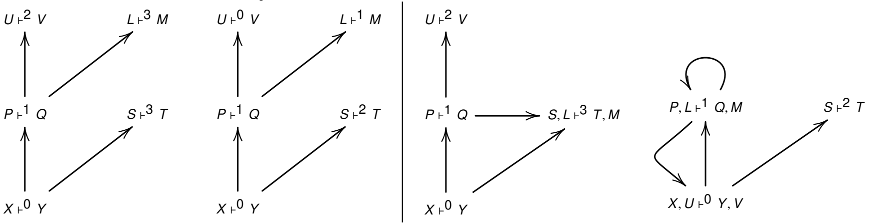
Indexed nested sequents (Fitting 2015)

► How can we extend the nested sequent framework to present cut-free calculi for more logics? What is the natural extension?

► Indexed nested sequents INS are nested sequents where each sequent is assigned a (not necessarily unique) index.

$$X \vdash^0 Y, [P \vdash^1 Q, [U \vdash^2 V], [L \vdash^3 M]], [S \vdash^3 T]$$
$$X \vdash^0 Y, [P \vdash^1 Q, [U \vdash^0 V], [L \vdash^1 M]], [S \vdash^2 T]$$

► Draw the trees, identify the nodes with the same index



Indexed nested sequent calculi for Geach (Lemmon-Scott) modal logics

- Geach logics extend normal modal logic K by Geach axioms:

$$G(h,i,j,k) := \Diamond^h \Box^i p \to \Box^j \Diamond^k p \quad (h,i,j,k \geq 0)$$

- E.g. $\Box p \to \Box\Box p$; $\Box p \to p$; $\Diamond\Box p \to \Box\Diamond p$
- Fitting's Geach scheme yields an INS rule corresponding to $G(h,i,j,k)$ when $i,j > 0$.
- Rule for $\Diamond\Box p \to \Box\Diamond p$. a,b,c are distinct indices. c not in conclusion.

$$\frac{\Gamma\{[X \vdash^a Y, \Delta, [\vdash^c]], [U \vdash^b V, \Sigma, [\vdash^c]]\}}{\Gamma\{[X \vdash^a Y, \Delta], [U \vdash^b V, \Sigma]\}}$$

- Fitting does not prove *syntactic cut-elimination*.
- Calculus minus the cut-rule is shown complete with respect to the semantics of the logic

How can we prove syntactic cut-elimination for INS reusing the proof for labelled sequents?

- Existing nested sequent calculi cut-elimination proofs are specialised, technical, intricate
- The labelled sequent calculus proof is generic (a single extension of Gentzen's proof works in all cases)
- Why create framework-specific proofs if we can reuse existing proofs?
- Recall: to adapt the LS proof we need to restrict *all* labelled sequents to trees where some nodes may be identified.

Inducing syntactic cut-elimination for indexed nested sequents

- Consider those labelled sequents where the *relation multiset* defines a tree:

$$\{\mathcal{R}, X \vdash Y \quad | \quad \mathcal{R} \text{ is a tree}\}$$

- This is what was used to relate NS and labelled tree sequents (LTS).
- To relate INS, let us add equality terms e.g. $\{x = y, u = z\}$ to the formal language. Call this a labelled tree sequent with equality (LTSE).

$$\{\mathcal{R}, \mathcal{E}, X \vdash Y \quad | \quad \mathcal{R} \text{ is a tree}\}$$

- (Negri 2005) shows that cut-elimination holds if $\mathcal{R}$ is not forced to be a tree.
- Notice that $\mathcal{E}$ can be compiled away if the LS are *not* restricted to trees.

Some details of the cut-elimination for LTSE calculi

$$\dfrac{\dfrac{\mathcal{R}, Rxy, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, y : A}{\mathcal{R}, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, x : \Box A}(\Box r) \qquad \dfrac{\mathcal{R}, Rxz, \mathcal{E}_2, x : \Box A, z : A \vdash \Gamma_2 \vdash \Delta_2}{\mathcal{R}, Rxz, \mathcal{E}_2, x : \Box A, \Gamma_2 \vdash \Delta_2}(\Box l)}{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\text{cut}$$

Then proceed:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}, Rxy, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, y : A}{\mathcal{R}, Rxy, Rxz, y = z, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, y : A}\text{weak}}{\mathcal{R}, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, z : A}\text{subs.} \qquad \text{cut}\left\{\dfrac{\begin{matrix}\mathcal{R}, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, x : \Box A \\ \mathcal{R}, Rxz, \mathcal{E}_2, x : \Box A, z : A \vdash \Gamma_2 \vdash \Delta_2\end{matrix}}{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_2, z : A \vdash \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\right.}{\dfrac{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_1, \mathcal{E}_2, \Gamma_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_1, \Delta_2}{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\text{ctr}}$$

- The premises of (cut) must have identical relation multiset $\mathcal{R}, Rxz$ (to preserve LTSE derivations)
- The dashed lines indicate proof transformations (see paper for details)
- Proof runs just as for LS (with specialised substitution lemma, resulting derivation is an LTSE-derivation by inspection)

Some details of the cut-elimination for LTSE calculi (cont.)

$$\dfrac{\dfrac{\mathcal{R}, Rxy, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, y : A}{\mathcal{R}, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, x : \Box A}(\Box r) \qquad \dfrac{\mathcal{R}, Rxz, \mathcal{E}_2, x : \Box A, z : A \vdash \Gamma_2 \vdash \Delta_2}{\mathcal{R}, Rxz, \mathcal{E}_2, x : \Box A, \Gamma_2 \vdash \Delta_2}(\Box l)}{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\text{cut}$$

Then proceed:

$$\dfrac{\dfrac{\dfrac{\mathcal{R}, Rxy, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, y : A}{\mathcal{R}, Rxy, Rxz, y = z, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, y : A}\text{weak}}{\mathcal{R}, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, z : A}\text{subs.} \qquad \text{cut}\left\{\dfrac{\begin{array}{c}\mathcal{R}, Rxz, \mathcal{E}_1, \Gamma_1 \vdash \Delta_1, x : \Box A \\ \mathcal{R}, Rxz, \mathcal{E}_2, x : \Box A, z : A \vdash \Gamma_2 \vdash \Delta_2\end{array}}{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_2, z : A \vdash \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\right.}{\dfrac{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_1, \mathcal{E}_2, \Gamma_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_1, \Delta_2}{\mathcal{R}, Rxz, \mathcal{E}_1, \mathcal{E}_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\text{ctr}}$$

- NOTE: subs. is a special instance of general substitution lemma for labelled sequent calculi.
- The general substitution lemma does not preserve LTSE-sequents
- The version here *does* preserve LTSE-sequents *and* is strong enough to obtain cut-elimination (LTS calculi did not permit such a substitution lemma!)

LTSE-calculi have cut-elimination. Soundness and completeness?

- Every rule of the LTSE-calculus can be viewed as a rule instance of the usual labelled sequent calculus. So soundness is immediate.

- Completeness: need to verify that all the axioms and rules of K + Geach axiom(s) is derivable. Actually, only the Geach axiom(s) require a (little) work.

- Here is the LTSE derivation of $\Diamond\Box p \to \Box\Diamond p$ using $\rho(1,1,1,1)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Rxy, Rxz, Ryu, Rzu, u = u', y : \Box p, \ldots, u' : p, u : p \vdash z : \Diamond p, \ldots, u' : p}{Rxy, Rxz, Ryu, Rzu, u = u', y : \Box p, \ldots, u : p \vdash z : \Diamond p, \ldots, u' : p}\text{Repl}}{Rxy, Rxz, Ryu, Rzu, u = u', y : \Box p, \ldots, u : p \vdash z : \Diamond p}\Diamond\text{r}}{Rxy, Rxz, Ryu, Rzu', u = u', y : \Box p \vdash z : \Diamond p}\Box\text{l}}{Rxy, Rxz, y : \Box p \vdash z : \Diamond p}\rho(1,1,1,1)}{Rxy, y : \Box p \vdash x : \Box\Diamond p}\Box\text{r}}{x : \Diamond\Box p \vdash x : \Box\Diamond p}\Diamond\text{l}}{\vdash x : \Diamond\Box p \to \Box\Diamond p}
$$

- By inspection: this is a LTSE-derivation. Generalises to *all* Geach axioms.

Some conclusions

- Don't reinvent the wheel. Syntactic cut-elimination is technical enough. See if it can be induced from existing work.
- Embedding a framework sheds light on the relative expressiveness of frameworks (with respect to presenting cut-free calculi for logics)
- To prove meta-theoretic results e.g. interpolation, decidability it helps to use the most 'economical framework' for the logic of interest
- Are (indexed) nested sequents 'more syntactic' than labelled sequents? No. The results here show that a subclass of the LS corresponds to the identical data structure as INS.
- The LTSE corresponds to a modal formula iff the INS corresponds to a formula
- It is not clear how to interpret an arbitrary LS as a formula. This is due to to the expressiveness of the framework.
- This is not to suggest that one framework should be thrown away in favour of the other. NS to INS is a bottom-up approach. LS to LTSE is a top-down approach.

Some more conclusions

- Can we use these insights to obtain new proof calculi? Yes!
- Previously, no (indexed) nested sequent calculi presented for logics between intuitionistic and classical logic
- However there are labelled sequent calculi for intermediate logics (Negri 2012).
- As before: suitable frame conditions can be translated into structural rules yielding cut-free calculi.
- Cut-elimination for LTSE-sequents is exactly as before
- Need to check that the intermediate axiom has an LTSE-derivation.
- E.g. $(p \to \bot) \vee ((p \to \bot) \to \bot)$ is derivable. So we obtain an indexed nested sequent calculus for $\mathbf{Ip} + (p \to \bot) \vee ((p \to \bot) \to \bot)$.