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Colors Make Theories Hard

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Vassily Kandinsky - Composition VII - 1913

Outline

Context, Problem and Goal

Context

"$\mathcal{T}$-solving": the satisfiability problem for conjunctions of quantifier-free literals for some first-order theory $\mathcal{T}$ of interest

- core problem in SMT and other disciplines
- widely and deeply investigated in the literature, for a variety of theories

General Question

Given $\mathcal{T}$:

- how can we establish [detect/prove] that $\mathcal{T}$-solving is NP-hard?
- how can we identify NP-hard fragments of $\mathcal{T}$?

Goal

A simple and general criterion for establishing the NP-hardness of $\mathcal{T}$-solving for theories with equality – or of their fragments

- a generalization to theories without equality

Background: The Graph k-Colorability Problem

Problem: k-colorability of a graph $\mathcal{G}$ (k fixed positive integer value)

Given an un-directed graph $\mathcal{G} \stackrel{\text{def}}{=} \langle \mathcal{V}, \mathcal{E} \rangle$ with vertexes $\mathcal{V} \stackrel{\text{def}}{=} \{V_1, ..., V_n\}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, plus a set of k distinct colors $\mathcal{C} \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$,
$\mathcal{G}$ is k-Colorable iff there exists a total map $color : \mathcal{V} \longmapsto \mathcal{C}$ s.t. $color(V_i) \neq color(V_{i'})$ for every $\langle V_i, V_{i'} \rangle \in \mathcal{E}$.

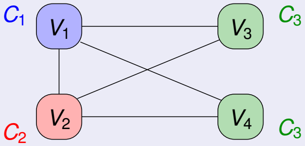
Complexity of k-colorability of a graph $\mathcal{G}$ [Karp, 1972]

- NP-complete if $k \geq 3$,
- in P otherwise.

From Graph k-Colorability to $\mathcal{T}$-solving: Intuitions

Example: encoding Graph k-Colorability into $\mathcal{LIA}$-solving

$\mathcal{G}$:



$$\bigwedge_{i=1}^{4}(\overbrace{(c_1 = 1) \wedge (c_2 = 2) \wedge (c_3 = 3) \wedge (v_i \geq 1) \wedge (v_i \leq 3)}^{\text{Colorer}_3[v_i, c_1, c_2, c_3]}) \wedge$$
$$\underbrace{\neg(v_1 = v_2) \wedge \neg(v_1 = v_3) \wedge \neg(v_1 = v_4) \wedge \neg(v_2 = v_3) \wedge \neg(v_2 = v_4) \wedge \neg(v_3 = v_4)}_{\text{Graph}_{[\mathcal{G}]}(v_1, \ldots, v_n)}$$

- Variables $\{v_1, ..., v_n\}$ and $\{c_1, ..., c_k\}$ represent vertexes and colors. Two parts:
  - $\text{Graph}_{[\mathcal{G}]}$: conjunction of negated equalities ($\mathcal{T}$-independent)
  - Colorer_k: expresses the fact that k distinct values $c_1, .., c_k$ exist and that v_i must be equal to one of $c_1, .., c_k$

$\Longrightarrow$ $\mathcal{T}$-solving for (this fragment of) $\mathcal{LIA}$ is NP-Hard

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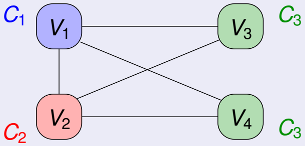

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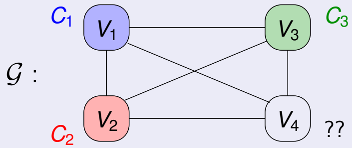

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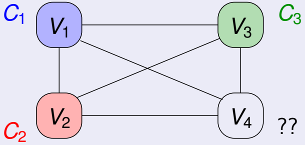

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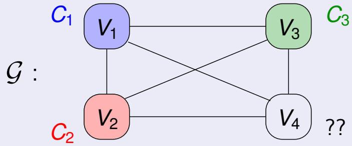

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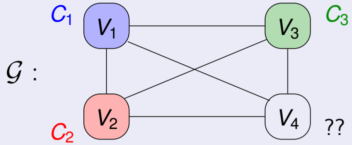
From Graph k-Colorability to $\mathcal{T}$-solving: Intuitions

Example: encoding Graph k-Colorability into $\mathcal{NLA}(\mathbb{R})$ -solving



$$\bigwedge_{i=1}^{4}\Big(\overbrace{(c_1 = 1) \wedge (c_2 = 2) \wedge (c_3 = 3) \wedge ((v_i - 1) \cdot (v_i - 2) \cdot (v_i - 3) = 0)}^{\text{Colorer}_3[v_i, c_1, c_2, c_3]}\Big) \wedge$$

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$\Longrightarrow$ $\mathcal{T}$-solving for (this fragment of) $\mathcal{NLA}(\mathbb{R})$ is NP-Hard

Outline

k-Colorers & k-Colorable Theories (with Equality)

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}}_i \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}}_i$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, [*i.e.*, $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k}(v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   a) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge (v_i = c_j)$
   b) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Basic Properties of a k-colorable theory $\mathcal{T}$

- $\text{AllDifferent}_k[\underline{\mathbf{c}}]$ is $\mathcal{T}$-satisfiable $\Longrightarrow$ domain size $\geq k$
- $\mathcal{T}$ is non-convex

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Example 1: Linear Integer arithmetic ($\mathcal{LIA}$)

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is a 3-Colorer for $\mathcal{LIA}$.

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We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Example 2: Non-Linear real arithmetic ($\mathcal{NLA}(\mathbb{R})$)

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Example 3: Arrays ($\mathcal{AR}$)

$\text{AllDifferent}_3[\underline{\mathbf{c}}] \wedge (A_2 = A_1\langle i_1 \leftarrow c_1 \rangle) \wedge$
$(A_3 = A_2\langle i_2 \leftarrow c_2 \rangle) \wedge (A_4 = A_3\langle i_3 \leftarrow c_3 \rangle) \wedge (v_i = A_4[i_1])$

is a 3-Colorer for $\mathcal{AR}$ (no interpreted constants!).

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Example 4: Bit-vectors of width 2 ($\mathcal{BV}_2$)

$\text{AllDifferent}_4[\underline{\mathbf{c}}]$
is a 4-Colorer for $\mathcal{BV}_2$.

Main results

Lemma

Let $\mathcal{G} \stackrel{\text{def}}{=} \langle \mathcal{V}, \mathcal{E} \rangle$ and $\mathcal{C} \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$ be respectively an un-directed graph with n vertexes $V_1, ..., V_n$ and a set of k distinct colors.
Let $\mathcal{T}$ be a k-colorable theory with equality, and let

$$\text{Enc}_{[\mathcal{G}\Rightarrow\mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n] \quad \stackrel{\text{def}}{=} \quad \bigwedge_{V_i \in \mathcal{V}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge \bigwedge_{\langle V_{i_1}, V_{i_2} \rangle \in \mathcal{E}} \neg(v_{i_1} = v_{i_2}),$$

Then $\mathcal{G}$ is k-colorable iff $\text{Enc}_{[\mathcal{G}\Rightarrow\mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n]$ is $\mathcal{T}$-satisfiable.

Theorem

If a theory with equality $\mathcal{T}$ is k-colorable for some fixed value $k \geq 3$, then the problem of deciding the $\mathcal{T}$-satisfiability of a quantifier-free conjunction of $\mathcal{T}$-literals is NP-hard.

To prove the NP-hardness of $\mathcal{T}$-solving, it suffices to find a k-colorer, for some $k \geq 3$

Outline

Exploiting finite domains of fixed size

$\mathcal{T}$ has finite domain of fixed size $k \geq 3$.
Then $\text{Colorer}_k[v_i, \underline{\mathbf{c}}] \stackrel{\text{def}}{=} \text{AllDifferent}_k[\underline{\mathbf{c}}]$.

Note: $(\{=, \neg =\})$ enough for NP-hardness!

Examples

$\mathcal{BV}_w$: All fragments of fixed-width Bit-Vector, with width $w > 1$

$\mathcal{FPA}_{e,s}$: All fragments of floating-point arithmetic
$e, s > 0$ size of exponent and significant respectively.
(E.g., $\mathcal{FPA}_{11,53}$ IEEE 754-2008 standard.)

...

Exploiting constants (e.g., Arithmetic)

$$\text{Colorer}_k[v_i, \mathbf{c}, \underline{\mathbf{y}_i}] \stackrel{\text{def}}{=} \bigwedge_{j=1}^{k}(c_j = t_j) \wedge \Psi[v_i, \underline{\mathbf{y}_i}],\ s.t.$$

- $\{t_1, ..., t_k\}$ interpreted constants represent k distinct domain values $\{D_1, ..., D_k\}$
- $\Psi[v_i, \underline{\mathbf{y}_i}]$ conjunction of literals admitting all and only the values $\{D_1, ..., D_k\}$ for v_i

Example 1. (fragment of) $\mathcal{LIA}$: Linear Integer arithmetic

$(c_1 = 1) \wedge (c_2 = 2) \wedge (c_3 = 3) \wedge (v_i \geq 1) \wedge (v_i \leq 3)$
is a 3-Colorer for $\mathcal{LIA}$.

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Example 2. (fragment of) $\mathcal{NLA}(\mathbb{R})$: Non-Linear real arithmetic

$(c_1 = 1) \wedge (c_2 = 2) \wedge (c_3 = 3) \wedge ((v_i - 1) \cdot (v_i - 2) \cdot (v_i - 3) = 0)$
is a 3-Colorer for $\mathcal{NLA}(\mathbb{R})$.

Exploiting closed terms (e.g., Lists)

$$\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}] \stackrel{\text{def}}{=} \bigwedge_{j=1}^{k} (c_j = t_j) \wedge \Psi[v_i, \underline{\mathbf{y}_i}], \; s.t.$$

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Example 3. $\mathcal{L}^+$: Theory of Lists s.t. $(\text{car}(\text{nil}) = \text{nil})$, $(\text{cdr}(\text{nil}) = \text{nil})$

$$\left(\begin{array}{l} (c_{11} = \text{cons}(\text{nil}, \text{nil})) \wedge (c_{21} = \text{cons}(\text{cons}(\text{nil}, \text{nil}), \text{nil})) \wedge \\ (c_{12} = \text{cons}(\text{nil}, \text{cons}(\text{nil}, \text{nil}))) \wedge (c_{22} = \text{cons}(\text{cons}(\text{nil}, \text{nil}), \text{cons}(\text{nil}, \text{nil}))) \wedge \\ \bigwedge_{i=1}^{2} ((\text{car}(x_i) = \text{car}(y_i)) \wedge (\text{cdr}(x_i) = \text{cdr}(y_i)) \wedge \neg(x_i = y_i)) \wedge \\ (v_i = \text{cons}(x_1, x_2)). \end{array} \right)$$

is a 4-Colorer for $\mathcal{L}^+$.

Hint [Nelson & Oppen, 80]:

$$((\text{car}(x_i) = \text{car}(y_i)) \wedge (\text{cdr}(x_i) = \text{cdr}(y_i)) \wedge \neg(x_i = y_i))$$
$$\models_{\mathcal{L}^+} (x_i = \text{nil}) \vee (x_i = \text{cons}(\text{nil}, \text{nil}))$$

General case: no constants (e.g., Finite Sets)

$$\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \stackrel{\text{def}}{=} \bigwedge_{j=1}^{k} (c_j = t_j[\underline{\mathbf{x}}_i]) \wedge \Phi[\underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \wedge \Psi[v_i, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i], s.t.$$

- $\{t_1[\underline{\mathbf{x}}_i], ..., t_k[\underline{\mathbf{x}}_i]\}$ generic terms on free variables $\underline{\mathbf{x}}_i$
- $\Phi[\underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{t}[\underline{\mathbf{x}}_i]}]$
- $\Psi[v_i, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k} (v_i = t_j[\underline{\mathbf{x}}_i])$
- $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i]$ verifies condition 3. a) and b).

Example 4. (fragment of) $\mathcal{S}$: Theory of Finite Sets

$$\left(\begin{array}{lll} (c_1 = \{x_1, x_2\}) & \wedge (c_2 = \{x_1\}) & \wedge \\ (c_3 = \{x_2\}) & \wedge (c_4 = \{\}) & \wedge \\ \neg(x_1 = x_2) & \wedge (v_i \subseteq \{x_1, x_2\}) & \end{array} \right).$$

is a 4-Colorer for $\mathcal{S}$.

Hint: exploit the 4 distinct subsets of $\{x_1, x_2\}$.

Note: no interpreted constants!

General case: no constants (e.g., Arrays)

$$\mathrm{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \stackrel{\text{def}}{=} \bigwedge_{j=1}^{k}(c_j = t_j[\underline{\mathbf{x}}_i]) \wedge \Phi[\underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i) \wedge \Psi[v_i, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i], s.t.$$

- $\{t_1[\underline{\mathbf{x}}_i], ..., t_k[\underline{\mathbf{x}}_i]\}$ generic terms on free variables $\underline{\mathbf{x}}_i$
- $\Phi[\underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \mathrm{AllDifferent}_k[\underline{\mathbf{t}[\underline{\mathbf{x}}_i]}]$
- $\Psi[v_i, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k}(v_i = t_j[\underline{\mathbf{x}}_i])$
- $\mathrm{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{x}}_i, \underline{\mathbf{y}}_i]$ verifies condition 3. a) and b).

Example 5. Arrays ($\mathcal{AR}$)

$\mathrm{AllDifferent}_3[\underline{\mathbf{c}}] \wedge (A_2 = A_1\langle i_1 \leftarrow c_1\rangle)\wedge$
$(A_3 = A_2\langle i_2 \leftarrow c_2\rangle) \wedge (A_4 = A_3\langle i_3 \leftarrow c_3\rangle) \wedge (v_i = A_4[i_1])$
is a 3-Colorer for $\mathcal{AR}$
Hint: exploit the implicit case-split of McCarthy's read-over-write array axioms:
if $i = j$ then $(A\langle i \leftarrow v\rangle[j] = v)$ else $(A\langle i \leftarrow v\rangle[j] = A[j])$
Note: no interpreted constants!

Outline

k-Colorers & k-Colorable Theories (without Equality)

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a generic theory, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}_i} \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}_i}$, which verify the following properties:

1. For every $\mathcal{T}$-intepretation $\mathcal{I}$, if $\mathcal{I} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}]$, then:

$$\text{for every } j, j' \in [1..k] \text{ s.t. } j \neq j', \qquad \langle c_j \rangle^{\mathcal{I}} \neq \langle c_{j'} \rangle^{\mathcal{I}},$$

2. For every $\mathcal{T}$-intepretation $\mathcal{I}$, if $\mathcal{I} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}]$, then:

$$\text{for some } j \in [1..k], \qquad \langle v_i \rangle^{\mathcal{I}} = \langle c_j \rangle^{\mathcal{I}}.$$

3. There exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ s.t.

   a) *for every* $j \in [1..k]$, $\quad \mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}]$ *and* $\langle v_i \rangle^{\mathcal{I}_{i,j}} = \langle c_j \rangle^{\mathcal{I}_{i,j}}$

   b) *for every* $j \in [1..k]$, $\quad \langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Theories which emulate equality/disequality

$\mathcal{T}$ emulates equality [disequality] iff exists a finite quantifier-free conjunction of $\mathcal{T}$-literals $\mathrm{Eq}[x_1, x_2]$ [$\mathrm{Neq}[x_1, x_2]$] such that, for every $\mathcal{T}$-interpretation $\mathcal{I}$,

$$\mathcal{I} \models_{\mathcal{T}} \mathrm{Eq}[x_1, x_2] \iff \langle x_1 \rangle^{\mathcal{I}} = \langle x_2 \rangle^{\mathcal{I}}.$$

$$[\ \mathcal{I} \models_{\mathcal{T}} \mathrm{Neq}[x_1, x_2] \iff \langle x_1 \rangle^{\mathcal{I}} \neq \langle x_2 \rangle^{\mathcal{I}}.\ ]$$

Lemma

Let $\mathcal{G} \stackrel{\text{def}}{=} \langle \mathcal{V}, \mathcal{E} \rangle$ and $\mathcal{C} \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$ (...).
Let $\mathcal{T}$ be a k-colorable theory which emulates equality and disequality. Let

$$\mathrm{Enc}_{[\mathcal{G} \Rightarrow \mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n] \stackrel{\text{def}}{=} \bigwedge_{v_i \in \mathcal{V}} \mathrm{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge \bigwedge_{\langle v_{i_1}, v_{i_2} \rangle \in \mathcal{E}} \mathrm{Neq}[v_{i_1}, v_{i_2}],$$

Then $\mathcal{G}$ is k-colorable iff $\mathrm{Enc}_{[\mathcal{G} \Rightarrow \mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n]$ is $\mathcal{T}$-satisfiable.

Theorem

If a theory $\mathcal{T}$ is k-colorable for some $k \geq 3$ and $\mathcal{T}$ emulates equality and disequality, then the problem of deciding the $\mathcal{T}$-satisfiability of a finite conjunction of quantifier-free $\mathcal{T}$-literals is NP-hard.

Theories which emulate equality/disequality

$\mathcal{T}$ emulates equality [disequality] iff exists a finite quantifier-free conjunction of $\mathcal{T}$-literals $\mathrm{Eq}[x_1, x_2]$ [$\mathrm{Neq}[x_1, x_2]$] such that, for every $\mathcal{T}$-interpretation $\mathcal{I}$,

$$\mathcal{I} \models_{\mathcal{T}} \mathrm{Eq}[x_1, x_2] \iff \langle x_1 \rangle^{\mathcal{I}} = \langle x_2 \rangle^{\mathcal{I}}.$$

$$[\, \mathcal{I} \models_{\mathcal{T}} \mathrm{Neq}[x_1, x_2] \iff \langle x_1 \rangle^{\mathcal{I}} \neq \langle x_2 \rangle^{\mathcal{I}}. \,]$$

Lemma

Let $\mathcal{G} \overset{\text{def}}{=} \langle \mathcal{V}, \mathcal{E} \rangle$ and $\mathcal{C} \overset{\text{def}}{=} \{C_1, ..., C_k\}$ (...).
Let $\mathcal{T}$ be a k-colorable theory which emulates equality and disequality. Let

$$\mathrm{Enc}_{[\mathcal{G} \Rightarrow \mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n] \overset{\text{def}}{=} \bigwedge_{v_i \in \mathcal{V}} \mathrm{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge \bigwedge_{\langle v_{i_1}, v_{i_2} \rangle \in \mathcal{E}} \mathrm{Neq}[v_{i_1}, v_{i_2}],$$

Then $\mathcal{G}$ is k-colorable iff $\mathrm{Enc}_{[\mathcal{G} \Rightarrow \mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n]$ is $\mathcal{T}$-satisfiable.

Theorem

If a theory $\mathcal{T}$ is k-colorable for some $k \geq 3$ and $\mathcal{T}$ emulates equality and disequality, then the problem of deciding the $\mathcal{T}$-satisfiability of a finite conjunction of quantifier-free $\mathcal{T}$-literals is NP-hard.

Theories which emulate equality/disequality

Example: Non-linear real arithmetic without equality

Let $\mathcal{NLA}(\mathbb{R})^{\setminus\{=\}}$ be the signature-restriction fragment of $\mathcal{NLA}(\mathbb{R})$ without equality. Let:

$$\mathsf{Eq}[x_1, x_2] \stackrel{\text{def}}{=} (x_1 \geq x_2) \wedge (x_2 \geq x_1)$$

$$\mathsf{Neq}[x_1, x_2] \stackrel{\text{def}}{=} ((x_1 - x_2) \cdot (x_1 - x_2) > 0).$$

Then:

$$\mathsf{Colorer}_3[v_i, c_1, c_2, c_3] \stackrel{\text{def}}{=} \mathsf{Eq}[c_1, 1] \wedge \mathsf{Eq}[c_2, 2] \wedge \mathsf{Eq}[c_3, 3] \wedge \mathsf{Eq}[(v_i - 1) \cdot (v_i - 2) \cdot (v_i - 3), 0].$$

is a 3-colorer for $\mathcal{NLA}(\mathbb{R})^{\setminus\{=\}}$.

Outline

Conclusions

- *k*-colorers & *k*-colorability: a simple and general method for establishing (identify/prove) the NP-hardness of $\mathcal{T}$-solving for theories with equality
  - allows for identifying NP-hard fragments
  - generalizes to theories without equality
- idea: produce a conjunction of literals expressing the fact that a variable assumes one among *k* distinct values
- proof-of-concept: simple NP-hardness proofs for (fragments of) known theories
- limitation: requires domain size ≥ 3

Ongoing & Future Work

Extensions & Generalizations

- adopt some more general notion of fragment
- extend $\mathrm{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$ to deal with uninterpreted function/predicate symbols
- enrich Eq[., .] and Neq[., .] with auxiliary variables, uninterpreted funs/preds

  $\Longrightarrow$ extend further the applicability to theories without equality
- extend $\mathrm{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$ to use tuples of variables $\underline{\mathbf{v}}_i$, $\underline{\mathbf{c}}_1, .., \underline{\mathbf{c}}_k$ instead of single variables to encode vertexes and colors (including ad hoc Neq[., .] functions).

  $\Longrightarrow$ overcome the restriction of domain size ≥ 3

Further Investigate Applicability

- extensive exploration of the pool of available NP-hard theories
  - pinpoint minimal NP-hard fragments
- explore the boundaries of k-colorability
  - looking for NP-hard theories which are not k-colorable

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EXTRA SLIDES

The following are extra slides, which I do not plan to present (unless explicitly asked for). They are intended for the participants's own convenience, in order to integrate the information of the presented slides.

Conjunctions of atoms wrt. conjunctions of literals

Lemma

Let $\mathcal{G} \stackrel{\text{def}}{=} \langle \mathcal{V}, \mathcal{E} \rangle$ and $\mathcal{C} \stackrel{\text{def}}{=} \{C_1, ..., C_k\}$ be respectively an un-directed graph with n vertexes $V_1, ..., V_n$ and a set of k distinct colors.
Let $\mathcal{T}$ be a k-colorable theory with equality, and let

$$\text{Enc}_{[\mathcal{G}\Rightarrow\mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n] \stackrel{\text{def}}{=} \bigwedge_{V_i \in \mathcal{V}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge \bigwedge_{\langle V_{i_1}, V_{i_2} \rangle \in \mathcal{E}} (v_{i_1} \neq v_{i_2})$$

Then $\mathcal{G}$ is k-colorable iff $\text{Enc}_{[\mathcal{G}\Rightarrow\mathcal{T}]}[v_1, ..., v_n, \underline{\mathbf{c}}, \underline{\mathbf{y}}_1, ..., \underline{\mathbf{y}}_n]$ is $\mathcal{T}$-satisfiable.

Remark

Let $\mathcal{T}$ admit atomic disequality predicates "$\neq$" ("$!=$", "<>",...) and a $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$ which is a set of $\mathcal{T}$-atoms, for some fixed value $k \geq 3$.
Then the problem of deciding the $\mathcal{T}$-satisfiability of a conjunction of quantifier-free $\mathcal{T}$-atoms is NP-hard.

Properties of k-colorable theories $\mathcal{T}$

Basic Properties of a k-colorable theory $\mathcal{T}$

- $\exists\underline{\mathbf{c}}.\ \text{AllDifferent}_k[\underline{\mathbf{c}}]$ is $\mathcal{T}$-valid $\Longrightarrow$ domain size $\geq k$
- $\mathcal{T}$ is non-convex

Property

If $\mathcal{T}'$ is a k-colorable theory with equality for some k, and $\mathcal{T}'$ is a signature-restriction fragment of another theory $\mathcal{T}$, then $\mathcal{T}$ is k-colorable.

Property

If $\mathcal{T}$ and $\mathcal{T}'$ are two signature-disjoint theories with equality and $\mathcal{T}$ is k-colorable for some k, then the combined theory $\mathcal{T} \cup \mathcal{T}'$ is k-colorable.

k-Colorers & k-Colorable Theories with Equality

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}}_i \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}}_i$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, $\quad$ [*i.e.,* $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k}(v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   a) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$, *and*
   b) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge (v_i = c_j)$.

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

k-Colorers & k-Colorable Theories with Equality

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}_i} \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}_i}$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, [*i.e.*, $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}] \models_{\mathcal{T}} \bigvee_{j=1}^{k} (v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   a) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$, *and*
   b) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}_i}] \wedge (v_i = c_j)$.

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Example 1: Linear Integer arithmetic ($\mathcal{LIA}$)

$(c_1 = 1) \wedge (c_2 = 2) \wedge (c_3 = 3) \wedge (v_i \geq 1) \wedge (v_i \leq 3)$
is a 3-Colorer for $\mathcal{LIA}$.

k-Colorers & k-Colorable Theories with Equality

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}}_i \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}}_i$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, $\quad$ [*i.e.*, $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k} (v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   a) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$, *and*
   b) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge (v_i = c_j)$.

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Example 2: Linear Integer arithmetic ($\mathcal{LIA}$)

$\text{AllDifferent}_3[\underline{\mathbf{c}}] \wedge \bigwedge_{j=1}^{3}((c_j \geq 1) \wedge (c_j \leq 3)) \wedge (v_i \geq 1) \wedge (v_i \leq 3)$
is another 3-Colorer for $\mathcal{LIA}$. (The c_i's are not assigned to interpreted constants!)

k-Colorers & k-Colorable Theories with Equality

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}}_i \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}}_i$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, $\quad$ [*i.e.*, $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k}(v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   a) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$, *and*
   b) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge (v_i = c_j)$.

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Example 3: Linear Integer arithmetic ($\mathcal{LIA}$)

$\text{AllDifferent}_3[\underline{\mathbf{c}}] \wedge \bigwedge_{j=1}^{3}((c_j \geq 1) \wedge (c_j \leq 3)) \wedge (v_i = 1)$
is not a 3-Colorer for $\mathcal{LIA}$. (Does not verify 3.*a*).)

k-Colorers & k-Colorable Theories with Equality

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}}_i \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}}_i$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, $\quad$ [*i.e.*, $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k}(v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   a) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$, *and*
   b) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge (v_i = c_j)$.

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Example 4: Non-Linear real arithmetic ($\mathcal{NLA}(\mathbb{R})$)

$(c_1 = 1) \wedge (c_2 = 2) \wedge (c_3 = 3) \wedge ((v_i - 1) \cdot (v_i - 2) \cdot (v_i - 3) = 0)$
is a 3-Colorer for $\mathcal{NLA}(\mathbb{R})$.

k-Colorers & k-Colorable Theories with Equality

Let $k \geq 3$ be some fixed integer value, $\mathcal{T}$ be a theory with equality, and v_i (vertex), $\underline{\mathbf{c}} \stackrel{\text{def}}{=} \{c_1, .., c_k\}$ (colors) and $\underline{\mathbf{y}}_i \stackrel{\text{def}}{=} \{y_{i1}, ..., y_{il}\}$ (auxiliary) be free variables.
We call k-colorer for $\mathcal{T}$, namely $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i]$, a finite conjunction of quantifier-free $\mathcal{T}$-literals with free variables v_i, $\underline{\mathbf{c}}$ and $\underline{\mathbf{y}}_i$, which verify the following properties:

1. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \text{AllDifferent}_k[\underline{\mathbf{c}}]$, $\quad$ [*i.e.*, $\bigwedge_{1 \leq j \leq j' \leq k} \neg(c_j = c_{j'})$]
2. $\text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \models_{\mathcal{T}} \bigvee_{j=1}^{k}(v_i = c_j)$,
3. there exist k $\mathcal{T}$-interpretations $\{\mathcal{I}_{i,1}, ..., \mathcal{I}_{i,k}\}$ *s.t.*
   - a) *for every* $j \in [1..k]$, $\langle c_j \rangle^{\mathcal{I}_{i,1}} = \langle c_j \rangle^{\mathcal{I}_{i,2}} = ... = \langle c_j \rangle^{\mathcal{I}_{i,k}}$, *and*
   - b) *for every* $j \in [1..k]$, $\mathcal{I}_{i,j} \models_{\mathcal{T}} \text{Colorer}_k[v_i, \underline{\mathbf{c}}, \underline{\mathbf{y}}_i] \wedge (v_i = c_j)$.

We say that $\mathcal{T}$ is k-colorable if and only if it has a k-colorer.

Example 5: Arrays ($\mathcal{AR}$)

$\text{AllDifferent}_3[\underline{\mathbf{c}}] \wedge (A_2 = A_1\langle i_1 \leftarrow c_1 \rangle) \wedge$
$(A_3 = A_2\langle i_2 \leftarrow c_2 \rangle) \wedge (A_4 = A_3\langle i_3 \leftarrow c_3 \rangle) \wedge (v_i = A_4[i_1])$

is a 3-Colorer for $\mathcal{AR}$ (no interpreted constants!).