Automated Discovery of Geometrical Theorems in GeoGebra

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Overview

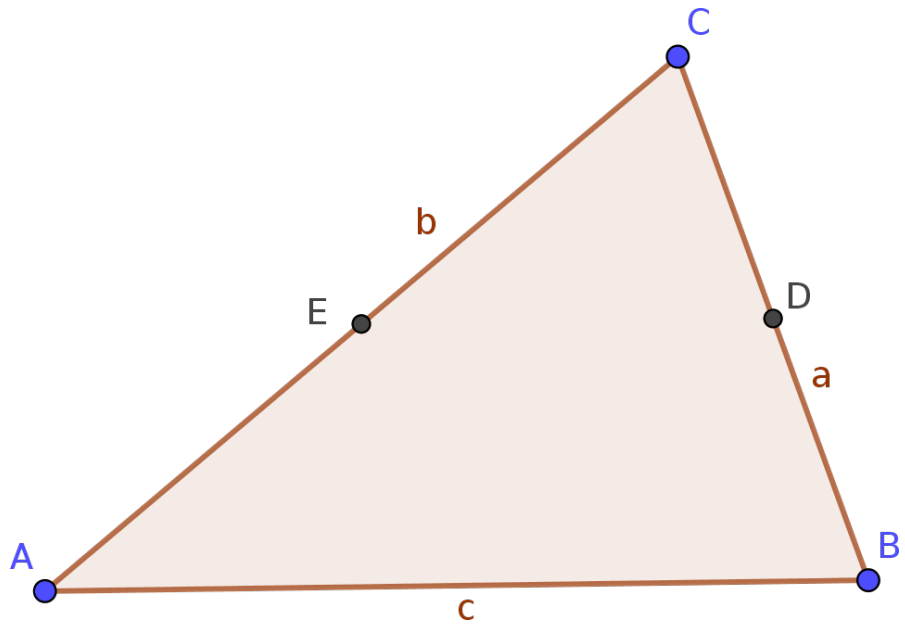
- **Introduction**
- Mathematical Background
- Applications to Theorems
- Discussion
- Conclusion

Introduction: GeoGebra and the Discover Command

- GeoGebra: Collection of open source, online geometry apps
- Discover Command
  - Our contribution to the GeoGebra software
  - Calculates relevant theorems and features pertaining to a planar Euclidean geometric figure
  - Source code at GitHub repository `http://github.com/kovzol/geogebra-discovery`

Introduction: Discover Command

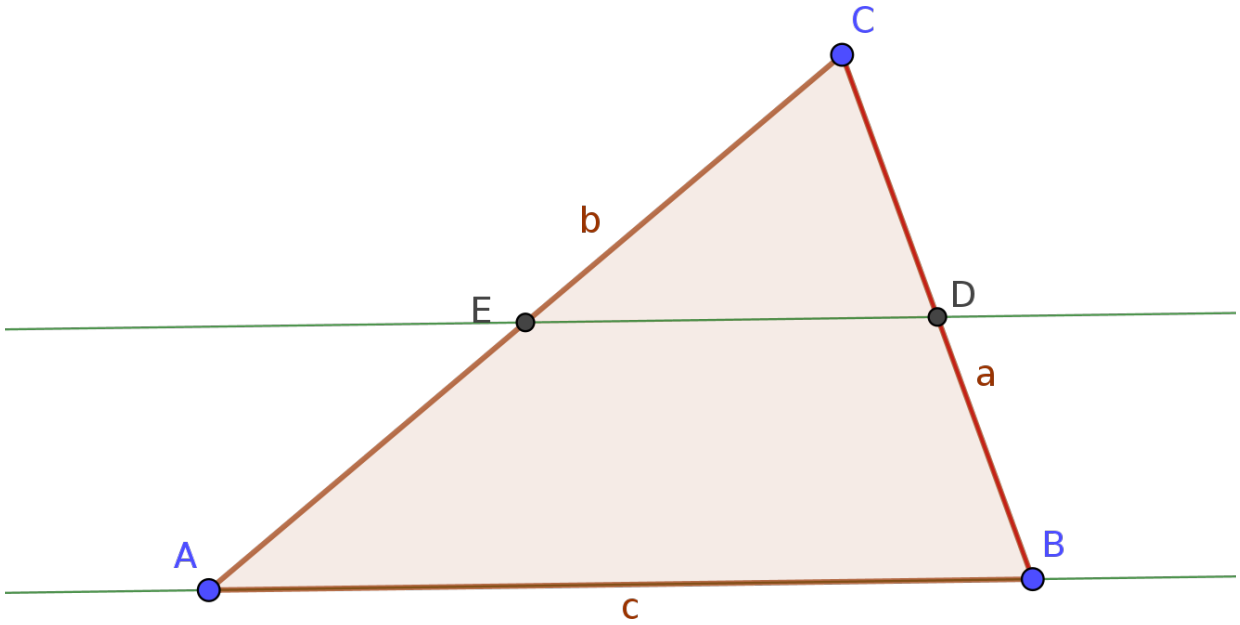
Midline theorem

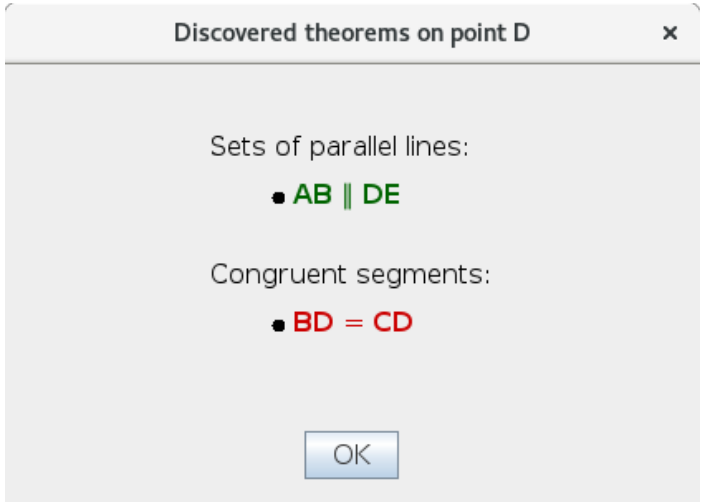


1. Let ABC be an arbitrary triangle.
2. Let D and E be the midpoints of BC and AC, respectively.
3. Execute command `Discover(`D`)`.

Introduction: Discover Command

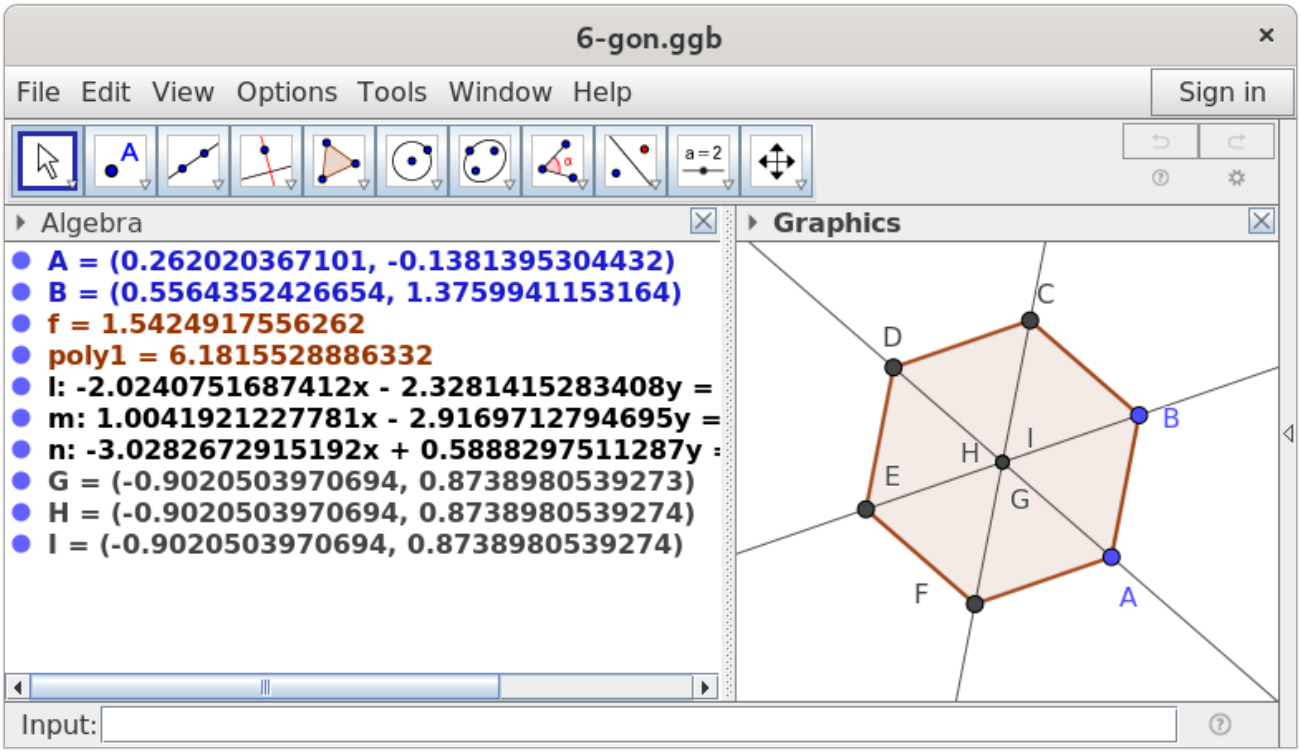
The `Discover` command reports the *Midline theorem*





Introduction: Discover Command

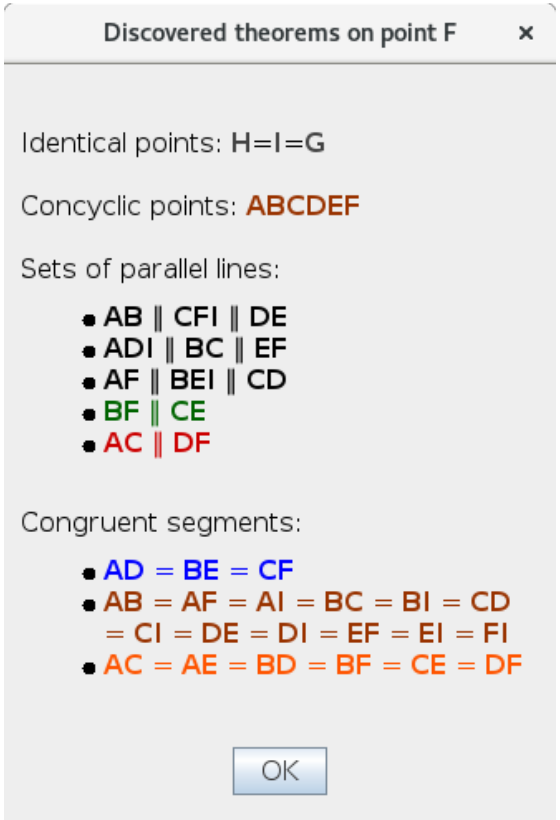
Exploration on a regular hexagon

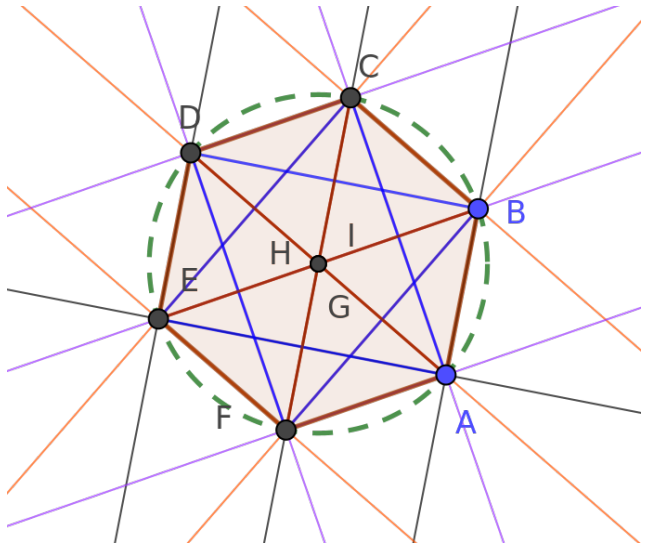


1. Regular hexagon $ABCDEF$
2. Point G is defined as the intersection of AD and BE
   $H = BE \cap CF$, $I = AD \cap CF$.
3. Execute Command `Discover(F)`.

Introduction: Discover Command

Several theorems related to point F, and the geometric output of discovery





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Mathematical Background

Overview of Discover algorithm:

1. Analyze all points to determine whether they are the same as another point.
2. Examine all possible point triplets for collinearity.
3. Check all possible 4-point subsets on the figure for concyclicity.
4. With knowledge of the collinear points, define separate lines and determine whether they are parallel.
5. Consider all possible point pairs and identify congruent segments.
6. Determine perpendicular lines.

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Applications to Theorems

Proved various theorems shown below

- Euler's Line
- Nine-point Circle
- An International Math Olympiad Problem

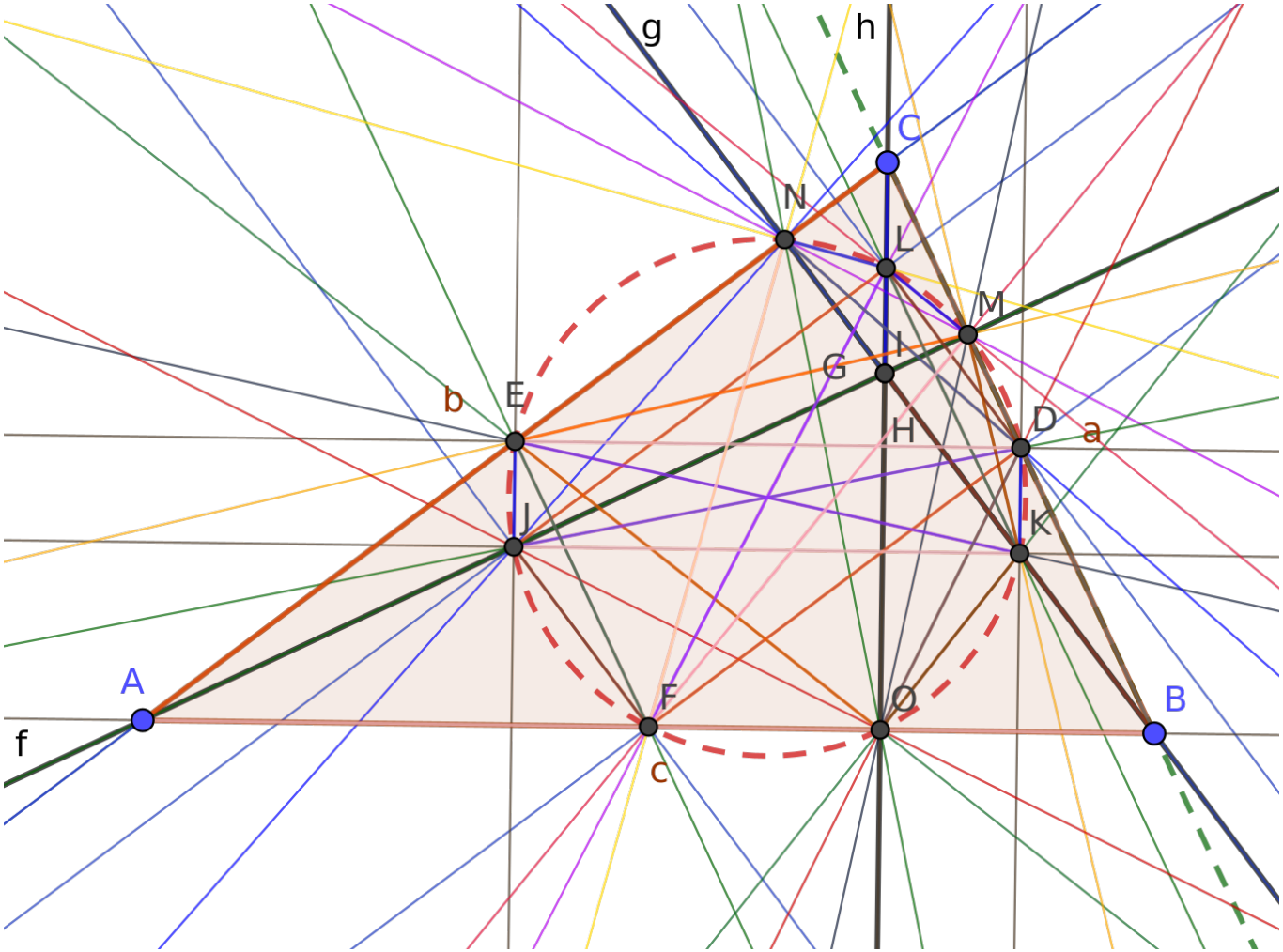
Applications to Theorems: Nine-Point Circle

Circle passing through 9 points of any triangle:

- Midpoint of each side of the triangle (Euler, 1765)
- Foot point of each altitude
- Midpoint of the line segment from each vertex of the triangle to the orthocenter (Feuerbach, 1822)

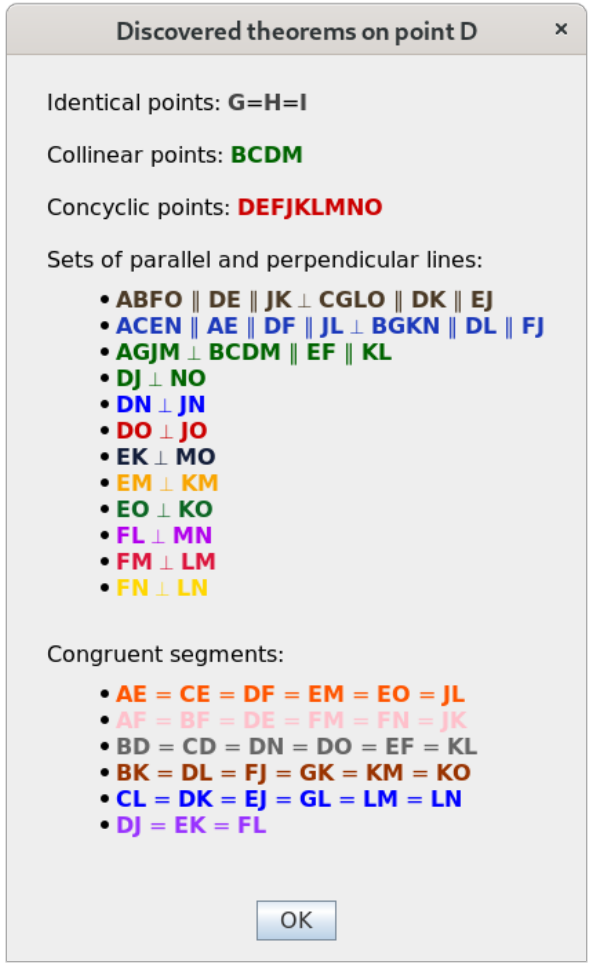
Applications to Theorems: Nine-Point Circle

Output of the command `Discover(D)`



Applications to Theorems: Nine-Point Circle

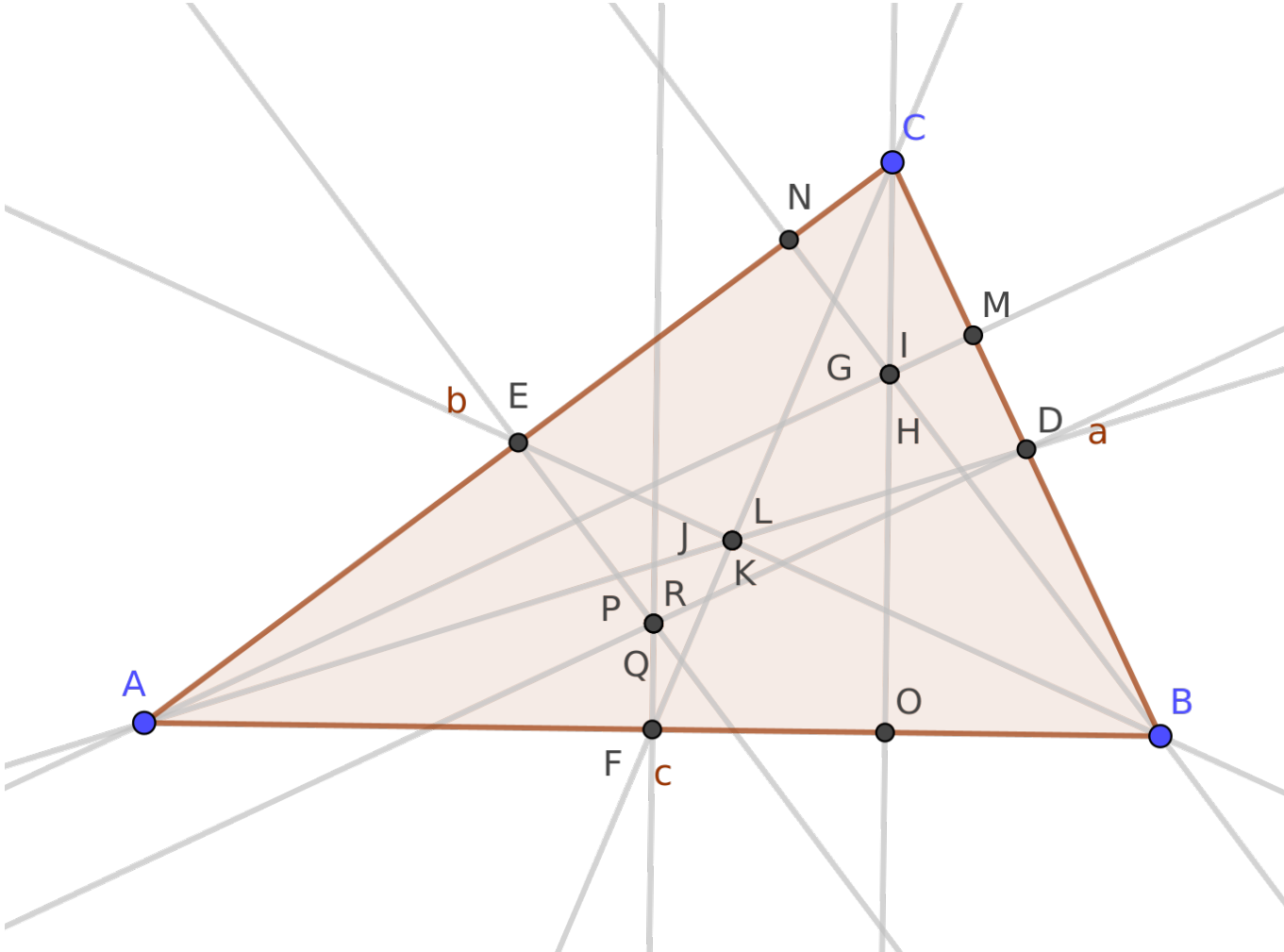
Output of the command `Discover(D)`



Applications to Theorems: Euler's Line

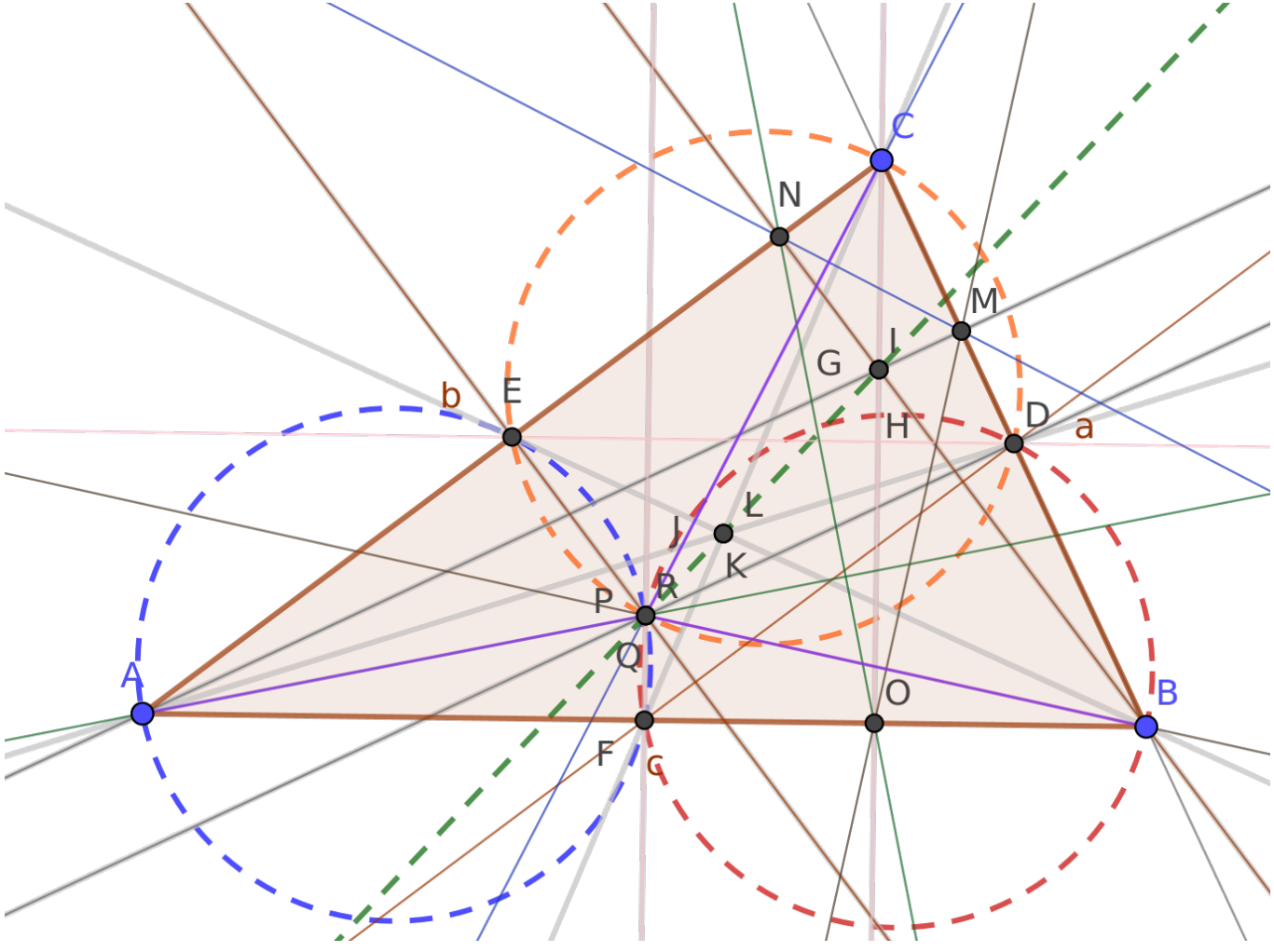
- Line determined in any triangle that is not equilateral
- Passes through orthocenter, centroid, and the circumcenter

Applications to Theorems: Euler's Line



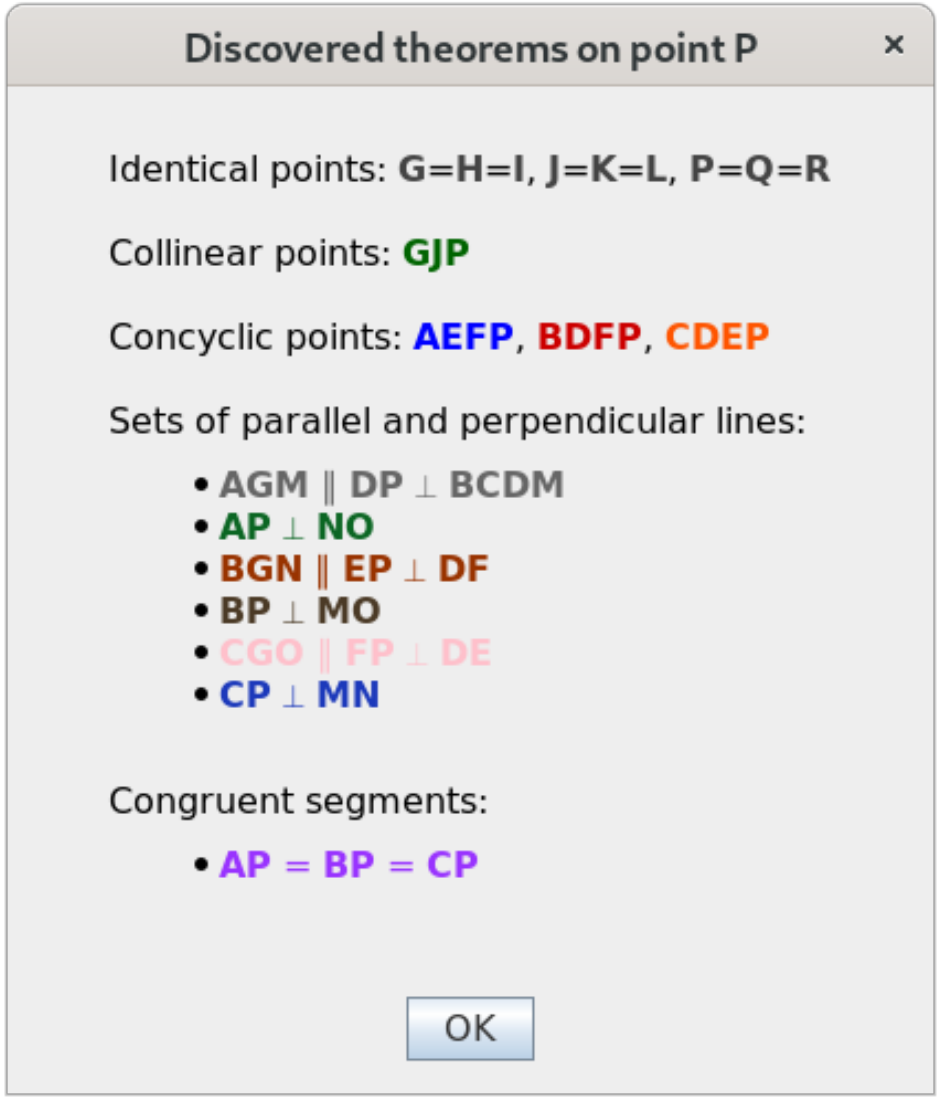
Applications to Theorems: Euler's Line

Output of the command `Discover(`P`)`



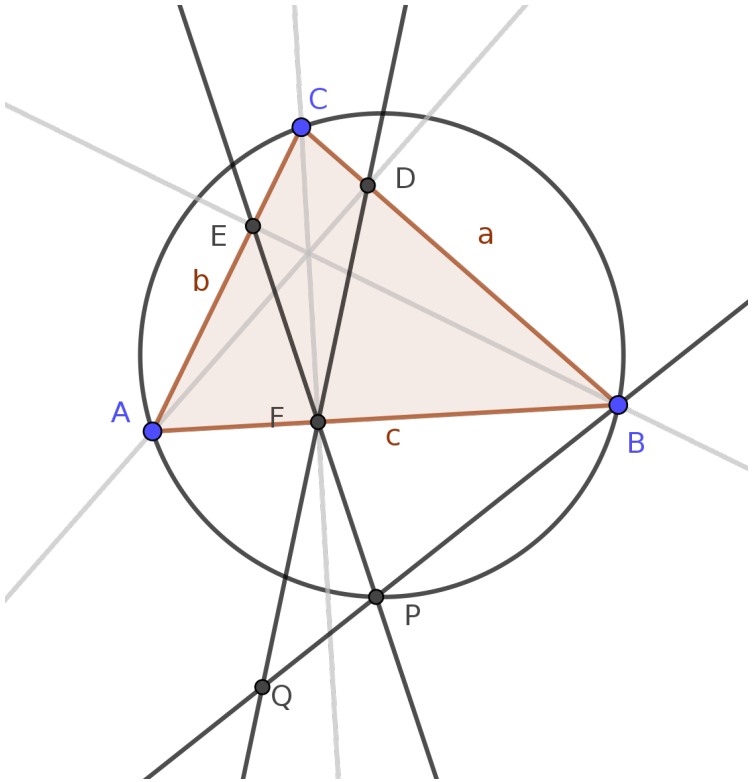
Applications to Theorems: Euler's Line

Output of the command `Discover(`P`)`



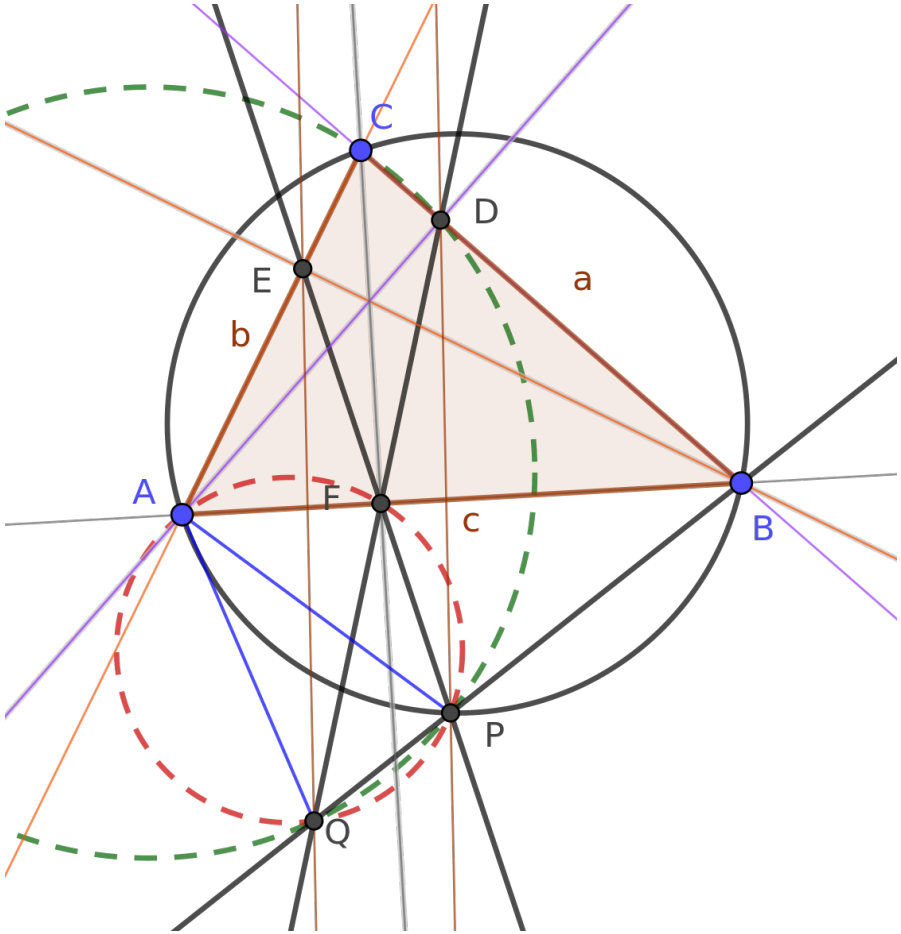
Applications: Int'l Math Olympiad problem

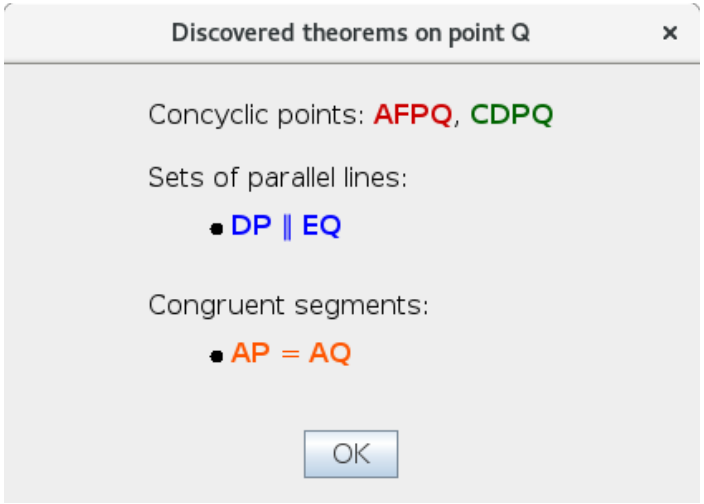
Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.



Applications: Int'l Math Olympiad

Output of the command `Discover(Q)`





Applications to Theorems

Other theorems

- Brahmagupta's theorem
- Napoleon's theorem
- Thebault's first two theorems

`https://prover-test.geogebra.org/job/GeoGebra_Discovery-discovertest/`

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Discussion

- Combinatorial explosion
- Related work

Discussion: Combinatorial explosion

- Computational complexity for proving one statement is at most doubly exponential in the number of variables.
- When dealing with more objects, the number of statements checked increases polynomially.
- The number of statements can be decreased by minimizing the number of objects (using equivalence relations and factor sets), and for each statement the number of variables can be decreased by fixing two points w.l.o.g.

Discussion: Related work

- Java Geometry Expert (Ye, Chou, Gao 2011)
- OK Geometry (Magajna 2019)

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Conclusion

- The Discover command integrates into the GeoGebra suite of open source math apps
- Performs rapid geometric analysis to determine important properties and theorems for the user's input
- Applications to education, automated discovery and exploration of geometric theorems

References

- Ye, Z., Chou, S.-C., Gao, X.-S. (2011) *An introduction to Java Geometry Expert.* In: Sturm T., Zengler C. (eds) Automated Deduction in Geometry. ADG 2008. Lecture Notes in Computer Science vol 6301. Berlin, Heidelberg: Springer. 189-195
- Magajna, Z. (2019) *Tools for automated hypothesising and proving in geometry classroom.* In: Slovak-Czech Conference on geometry and graphics: proceedings. Bratislava: Vydavatel'stvo SCHK. 19-24
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