Visual Geometry Proofs in a Learning Context

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1 Introduction

Geometry with its formal, logical and spatial properties is well suited to be taught in an environment that includes dynamic geometry software (DGSs), geometry automated theorem provers (GATPs) and repositories of geometric problems (RGPs). With the integration of those tools in a given learning environment the student is able to explore the built-in knowledge, but also to do new constructions, and test new conjectures. In such an environment the student can visualise geometric objects and link the formal, axiomatic, nature of geometry with its standard models and corresponding illustrations, e.g., Euclidean Geometry and the Cartesian model. With the help of a geometry automated theorem prover it is possible to check the soundness of the constructions, e.g. if two given lines are parallel, and also to make formal proofs of geometric conjectures.

In proceedings of the ICMI Study 19 Conference: *Proof and Proving in Mathematics Education* [11, 12] we can find many articles exploring the use of proofs in a learning environment. In [6, 7] Gila Hanna gave the following useful list of the usefulness of proofs and proving in a learning environment:

- verification (concerned with the truth of a statement);
- explanation (providing insight into why it is true);
- systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems);
- discovery (the discovery or invention of new results);
- communication (the transmission of mathematical knowledge);
- construction of an empirical theory;
- exploration of the meaning of a definition or the consequences of an assumption;
- incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective.

She also state that "the best proof is one that also helps understand the meaning of the theorem being proved: to see not only that it is true, but also why it is true. Of course such a proof is also more convincing and more likely to lead to further discoveries." In the classroom, the fundamental question that a proof must address is surely 'why?'. In an educational domain, then, it is only natural to view proof first and foremost as explanation, and in consequence to value most highly those proofs which best help to explain. It happens that geometry enjoys a special position given the fact that most of its proofs are explanatory.

Dynamic software has the potential to encourage both exploration and proof, because it makes it so easy to pose and test conjectures, the property preserving manipulations allows the student to explore 'visual proofs' of geometric conjectures. Such a powerful feature provides the student with strong evidence that the theorem is true (and reinforces the value of exploration in general in giving students confidence in a theorem).

The teacher's classroom challenge is to exploit the excitement and enjoyment of exploration to motivate students, still to be able to explain that a visual exploration is not a proof, it can be used as a useful aid, but it is still only the exploration of a finite number of cases. One reason to go an extra step, supplying a proof to the students, is that exploration does not reflect the totality of mathematics itself, because mathematicians aspire to a degree of certainty that can only be achieved by proof. A second reason is that students should come to understand the first reason: As most mathematics educators would still agree, students need to be taught that exploration, useful as it may be in formulating and testing conjectures, does not constitute proof [6, 7].

Geometry automated theorem provers open the possibility to a formal validation of properties of given geometric constructions [10]. Apart the work done in the GCLC[1] the new version of GeoGebra [8] already includes a connection to GATPs allowing to give a formal answer to a given validation question [2].

Another important addition to any learning environment would be a GATP with the capability of readable formal proofs, human-readable and/or visual counterparts [4, 5, 9, 18, 17].

In automated theorem proving in geometry has two major lines of research: synthetic proof style and algebraic proof style [3, 19]. Algebraic proof style methods are based on reducing geometric properties to algebraic properties expressed in terms of Cartesian coordinates. These methods are usually very efficient, but the proofs they produce do not reflect the geometrical nature of the problem and they give only yes or no conclusion. Synthetic methods attempt to automate traditional geometry proof methods, producing human-readable and, because of that, they are more suited to be used in a learning environment.

A long term goal of the Web Geometry Laboratory project, an adaptive and collaborative blended-learning Web-environment, integrating a dynamic geometry system [14][2] is to include GATPs with the capability of having a human-readable or even visual counterpart to the formal proofs.

In the next sections we speak about the possibility to have formal proofs with a visual support.

2 Area Method Visual Proofs

The *area method* and the *full-angle method* are two semi-synthetic methods providing human-readable proofs [4, 5, 9].

The following simple example briefly illustrates some key features of the *area method*.

Example 2.1 (Ceva's Theorem) *Let $\triangle ABC$ be a triangle and P be an arbitrary point in the plane. Let D be the intersection of AP and BC, E be the intersection of BP and AC, and F the intersection of CP and AB. Then:*

$$\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = 1$$

[1] `http://poincare.matf.bg.ac.rs/~janicic/gclc/`

[2] Santos, Vanda and Quaresma, Pedro and Marić, Milena and Campos, Helena, *Web Geometry Laboratory: Case Studies in Portugal and Serbia*, submitted to Educational Technology Research and Development, May 2015.

The points A, B, C, and P are *free points*, points not defined by construction steps. The point D is the intersection of the line determined by the points A and P and of the line determined by the points B and C. The points E and F are constructed in a similar fashion.

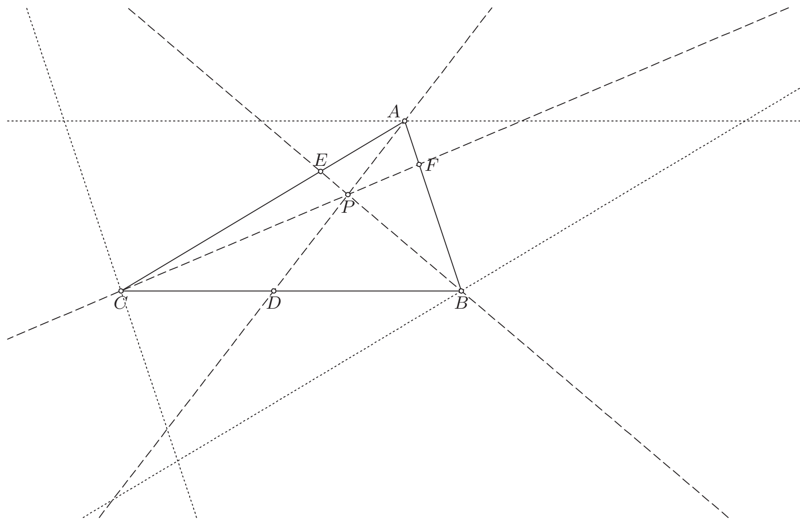

Figure 1: Illustration for Ceva's theorem

For stating and proving conjectures, the *area method* uses a set of specific *geometric quantities* that enable treating arrangement relations. The *ratio of parallel directed segments* ($\frac{\overline{AB}}{\overline{CD}}$) the *signed area* ($\mathscr{S}_{ABC}$) and the *Pythagoras difference* ($\mathscr{P}_{ABC}$).

The proof of a conjecture is based on eliminating all the constructed points, in reverse order, using for that purpose the properties of the geometric quantities, until an equality in only the free points is reached.

It can be proved that $\frac{\overline{AF}}{\overline{FB}} = \frac{\mathscr{S}_{APC}}{\mathscr{S}_{BCP}}$. By analogy $\frac{\overline{BD}}{\overline{DC}} = \frac{\mathscr{S}_{BPA}}{\mathscr{S}_{CAP}}$ and $\frac{\overline{CE}}{\overline{EA}} = \frac{\mathscr{S}_{CPB}}{\mathscr{S}_{ABP}}$. Therefore:

$$\begin{aligned}
\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} &= \frac{\mathscr{S}_{APC}}{\mathscr{S}_{BCP}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} && \text{the point } F \text{ is eliminated}\\
&= \frac{\mathscr{S}_{APC}}{\mathscr{S}_{BCP}}\frac{\mathscr{S}_{BPA}}{\mathscr{S}_{CAP}}\frac{\overline{CE}}{\overline{EA}} && \text{the point } D \text{ is eliminated}\\
&= \frac{\mathscr{S}_{APC}}{\mathscr{S}_{BCP}}\frac{\mathscr{S}_{BPA}}{\mathscr{S}_{CAP}}\frac{\mathscr{S}_{CPB}}{\mathscr{S}_{ABP}} && \text{the point } E \text{ is eliminated}\\
&= 1
\end{aligned}$$

Q.E.D.

The example illustrates how to express a problem using the given geometric quantities and how to prove it, and moreover, how to give a proof that is concise and very easy to understand.

Is it possible to have a visual reading of that proof? We think so, the following pictures (see Figures 2 to 4) have a direct connection to the formal proof and could be used to allow the illustration of the proof, for a better understanding, for a improved learning aid.

For the *area method* there are, as far as the authors know, no system providing such a connection. For the *full-angle method* the *Java Geometry Expert* (*JGEX*[3]) [20, 21, 22] provides such a connection.

[3] `http://www.cs.wichita.edu/~ye/`

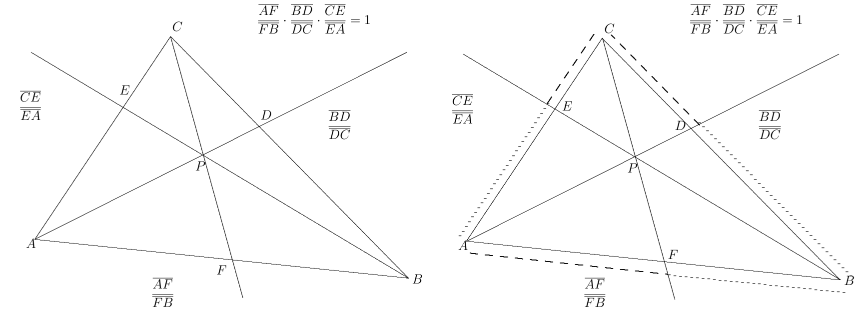

Figure 2: Ceva's Theorem – Visual Proof, Step 1 & 2

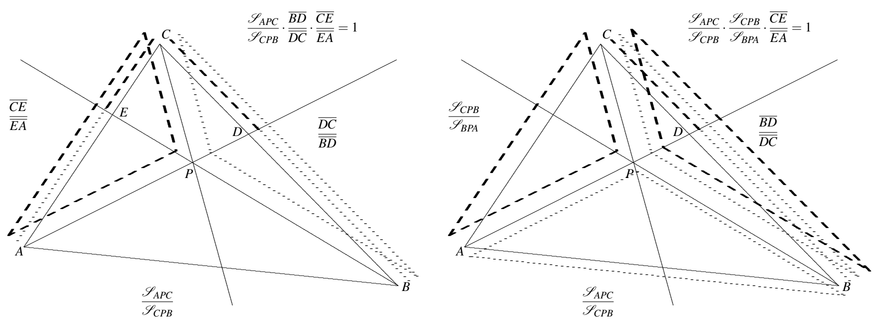

Figure 3: Ceva's Theorem – Visual Proof, Step 3 & 4

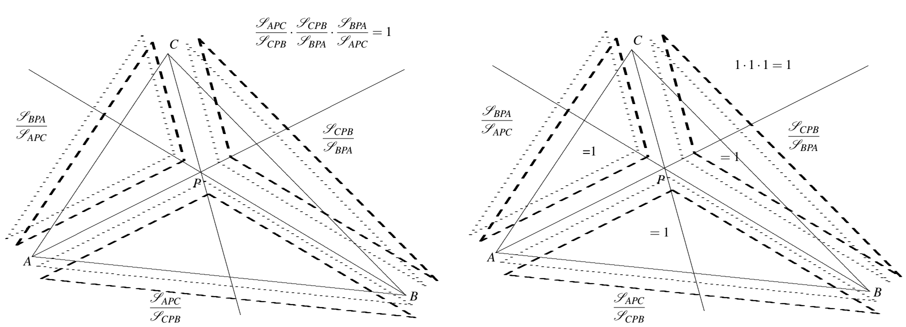

Figure 4: Ceva's Theorem – Visual Proof, Step 5 & 6

3 Full Angle Method Visual Proofs

Using the *JGEX* system we can build a given construction, state a conjecture about it and then, using one of the built-in GATPs, prove it. Using the *full-angle method* based GATP we can produce examples where the formal proof has a visual counterpart.

The figures 5 and 6 where taken from the tool own set of examples. When 'clicking' on a given step of the formal proof, a visual animation of the step is given on the construction. At first the related relations between objects of the construction, e.g. the angles between two lines in figure 5 are shown 'blinking', then the became fixed but using colours to clearly showing the corresponding relations being established in the formal proof.

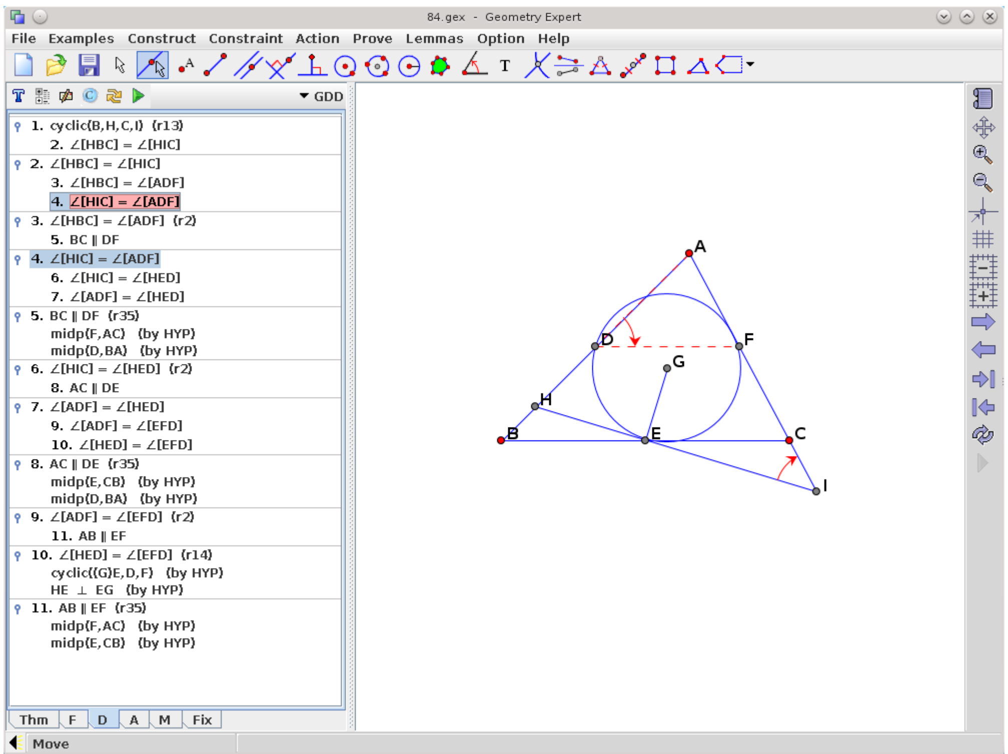

Figure 5: JGEX – Example 84, Step 2

In both this situations the drawback is that neither the *area method* nor the *full-angle method* used the usual set of axioms and rules of inference of secondary school geometry. They use the geometric quantities *ratio of parallel directed segments*, *signed area*, *Pythagoras difference* and *full angle*, and the axioms and rules of inference for those geometric quantities. The use of those methods in secondary school could be proved difficult.

4 Hybrid Language

Another interesting approach that we are exploring is the construction of a controlled hybrid language for geometry, a pair of controlled languages (natural and visual) with common semantics. By considering figures as sentences in a visual language sharing semantics with the natural language of geometric state-

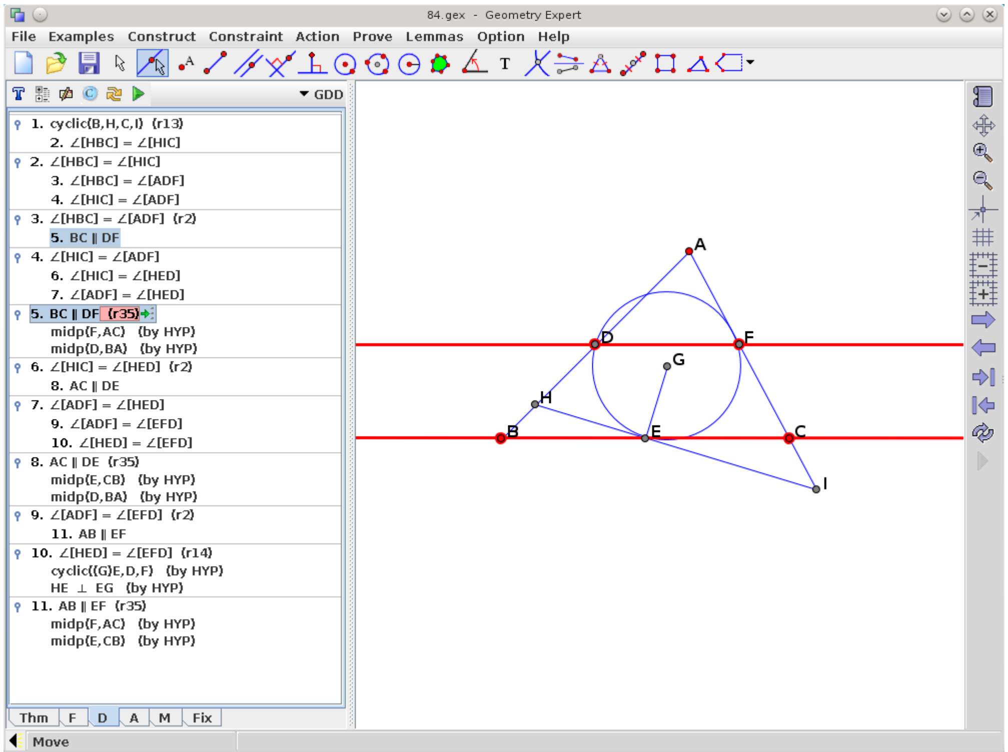

Figure 6: JGEX – Example 84, Step 5

ment, we can get interaction between parts of text and corresponding figures, connecting formal proofs to natural languages description, to visual descriptions. This last approach would be more generic then the concrete cases described above. This is a research line being already pursued by one of the authors along with Yannis Haralambous.[4]

5 Conclusions & Future Work

Geometry with its very strong and appealing visual contents and its also strong and appealing connection between the visual content and its formal specification, is an area where computational tools can enhance, in a significant way, the learning environments.

Dynamic geometry software systems significantly help students to acquire knowledge about geometric objects and, more generally, to acquire mathematical rigour. The geometry automated theorem provers capable of construction validation and production of human readable proofs, will consolidate the knowledge acquired with the use of the DGSs. If the GATP produces synthetic proofs, the proof of a conjecture or the proof of soundness of a construction can be used as an object of study, providing a logical explanation. We claim that the GATPs can be used in the learning process [10, 13, 15, 16].

The *full-angle method* implementation within *Open Geo Prover*[5] is part of the *Web Geometry Laboratory* (*WGL*) project [1]. The *Web Geometry Laboratory* is already a collaborative and adaptive blended

[4] Haralambous, Yannis and Quaresma, Pedro, *Geometric Statements as Controlled Hybrid Language Sentences, an Example*, submitted to Mathematics in Computer Science, July 2015.

[5] `https://code.google.com/p/open-geo-prover/`

learning platform being used in Portugal and Serbia. The integration of GATPs into the *Web Geometry Laboratory* will allow students to explore the connection between the visual content and its formal specification, consolidating the geometric knowledge of the students [15, 16].

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