Creating Usable Computer Tools that Reason Mathematically

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This paper discusses approaches taken to support usability in the open source Mathtoys system, and the rationales for them. Mathtoys has been designed and built to support a rather general audience of non-specialist users in applying correct reasoning to solve mathematical problems, with high school-level algebra as its starting point.

1 Introduction

Mathematics today is fundamental to science, technology, engineering, finance, and commerce. Among the consequences, sophisticated computer tools based on deductive reasoning are growing in capability and acceptance among sophisticated specialist users.

Mathematical reasoning also continues to be recognized as an important aspect of mathematics study around the world. For example, in the Common Core State Standards for Mathematics [3], adopted in 43 of the 50 U. S. states, three of the eight overall standards for mathematical practice relate specifically to reasoning.

Mathtoys is a project with the goal of helping to spark a growing movement to build and explore human-friendly computer tools that reason mathematically, starting with high school algebra.

But is this too ambitious? Does it mean asking students to learn advanced concepts to solve their basic problems? Mathtoys is based on the belief that it is not asking too much, and that students of math will find their effort well rewarded. Among its important benefits, an interactive computer tool can speed work, prevent errors, and guide the users problem-solving choices.

Certainly the typical early adopters will be energetic and inquisitive people with an interest in learning more about math. These same types will also tend to become the power users, learning much more about the tool, how it works and what it can do. A tool used just by these could be called a success. But successful adoption tends to spread, and I believe such a tool can also benefit the greater numbers of students who are just following a curriculum. It can open doors into mathematics not just for those who will explore deeply, but also for those who just want to fetch what they need and move on.

2 Reasoning in high school algebra

Mathtoys applies deduction to the learning of textbook mathematics at the high school level, starting with algebra. There is logical reasoning hiding inside high school algebra, and there are proofs just under the surface of solutions to algebra problems. Each problem-solving recipe has a certain proof structure behind it.

Viewed this way, problem-solving steps that are treated specially in textbooks become opportunities to see how different situations call for proofs with slightly different structure. For example, when passing from $x = y$ to $x^2 = y^2$, which can lead to extraneous candidate solutions. Partial functions including division by zero, inverses of functions, and such provide additional opportunities. Problems with more than

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expected solutions or none provide further opportunities to connect algebra with mathematical reasoning. For example a solution of false indicates there is no solution.

Equation solving as proof. A problem to solve an equation is presented to students as an equation E. Simultaneous equations are presented as a list, E1, ... , En. The variables have intended types, but this is usually implicit. In basic they are usually real numbers.

Writing R for the conjunction of the type restrictions, P for the conjunction of the equations, and S for the solution statement, the solution is a statement of the form $R \Rightarrow (P \Leftrightarrow S)$. So equation solving is a form of simplification, with the expression to simplify being an equation or conjoined set of equations rather than a term. The classic form of a solution equates a variable with a term that has no variables; for multiple variables, a conjunction, and for a finite set of solutions, a disjunction.

Partial functions such as square root and division by zero complicate the story a little bit. Support for these is in development, based on defining the values of such functions to be outside the range of the function in cases where the inputs are outside the domain. This allows use of conventional logic, which only supports total functions, as opposed to a partial logic or free logic. Problem statements implicitly require values of all expressions to be within the proper range of functions and operators applied to them. A discussion of various approaches to partial functions can be found in [4].

3 Approaching usability

Mathtoys is targeted toward developing skills and understanding of algebra in particular, and through algebra, mathematics and mathematical reasoning generally.

Getting users started. Mathtoys is Web-based, which helps make access easy. The Mathtoys Web site offers sample problems, so the user can get started without learning to enter a problem as a set of assumptions. The user normally selects a next step from a list of choices, reducing the need to learn commands.

Proof presentation. The customary presentation of many math problems is a list of formulas. Mathtoys presents of a typical equation-solving problem shows the equations in the problem, one per line, each preceded by its line number in the proof. In a line like this:

[1] 1: $8 \cdot (2 - t) = -5 \cdot t$ *assumption*

the [1] means that this is line 1 of the proof. The rest of the line represents the statement that $8 \cdot (2-t) = -5 \cdot t \Rightarrow 8 \cdot (2-t) = -5 \cdot t$. The use of "1:" is an example of the general convention that a proof line preceded by a line number represents a statement introduced on that line. Proof steps use the same format, replacing the word assumption with information about the inference applied.

Inference steps. The characteristic inference steps in textbook math operate directly on equations. In Mathtoys the foundation of most inference steps is replacement of an expression with one already shown to be equal to it, merging assumptions from the two steps. It underlies application of basic laws such as distributivity, applying the same operation to both sides of an equation, adding or subtracting equations, and moving terms.

The types of reasoning facilitated by Mathtoys may result in an appearance similar to what is sometimes called a "calculational" proof style. Mathtoys also has presentational devices that can increase this similarity.

Selecting expressions. Many inference steps used in textbook math apply to a specific subexpression. The Mathtoys GUI emphasizes support for selecting expressions. Moving the mouse over a formula causes selectable expressions to highlight as the mouse moves over them, serving as hints to the user.

Building proofs interactively. A user working on an algebra problem in Mathtoys normally proceeds by selecting an existing step or an expression within a step. Mathtoys displays a set of choices for operating on the step or using the selected expression. The user chooses a next step and the process repeats. Some proof steps take extra input, but most only need the selection as input. If the latest steps do not look promising, the user can delete them. When a step has the classic form of a solution, the user interface reports this to the user.

When the user is solving an algebra problem, Mathtoys by default offers only a subset of the kinds of steps that can be applied in algebra problems. The user can ask Mathtoys to offer more kinds of reasoning steps, but many kinds of steps not offered to the user by default are applicable much more often than they are actually useful. For example, it is very often legitimate to convert an expression e to $1 \cdot e$ using the identity $x = 1 \cdot x$. But this is not as often useful as an explicit solution step.

Simplification. After most steps, Mathtoys attempts to apply basic simplifications. If simplifications apply, it displays the result of the simplifications as an additional step.

Exploring proofs. Mathtoys carries out all proofs, with each step recording the type of inference done and its inputs. Higher-level inference steps also record their lower-level internal inferences, and the user interface can display any part at any level of detail through the user interface. The display of a step indicates the line numbers of any steps it depends on, other than the one just before it

4 Comparison with related work

While Mathtoys can be considered fundamentally as a variety of proof assistant, it does not support tactics or other top-down styles of proof, and presents itself as a tool for algebra problem-solving rather than theorem proving.

Mathtoys shares much of its point of view with Michael Beeson's MathPert, as admirably described in papers such as [2]. Many of these principles relate specifically to usability. Compared with MathPert however, Mathtoys expects somewhat more self-direction on the part of the user, in tune with its target audience of more adventurous early adopters. Specifically, where MathPert is designed to give the user hints through its "auto mode", there is no counterpart in Mathtoys. Architecturally though a similar approach could be applied to Mathtoys.

Again compared with MathPert, Mathtoys is more explicit in representing the reasoning in each step. Mathtoys sometimes presents logical symbols to a user working on an algebra problem, and tends to present them much more freely in displays of the details behind problem-solving steps.

5 Project status

Mathtoys is an open source project implemented in JavaScript and running in standard Web browsers. All inference as well as the user interface are implemented in JavaScript. The project Web site is `http://mathtoys.org/`.

The logical foundation currently follows Peter Andrews Q_0 formulation of Church-style type theory [1], treating the real numbers as a subset of the individuals. As currently used in Mathtoys however, it could as well be implemented in first-order logic with axioms for the real numbers, using substitutivity of equality and equivalence as basic inference rules.

The basic inference engine and the user interface facilities are the most developed aspects of Mathtoys. Mathematics of real numbers, including automatic simplification and the library of useful algebra facts is much more in development.

Near term work in progress includes more automation of simplification, more high-level problem-solving steps, and support for solving equations that use partial functions, including division by zero. The tool also needs to guide the user to demonstrate that derived solutions are fully equivalent to their problem statements, with appropriate levels of automation. Extending the math covered to include additional topics such as rational functions, exponentials, logarithms, and trigonometric functions will also be important to success as a practical educational tool. Another goal is to support proof of actual theorems, as presented for example in [5].

Providing suitable degrees of automation to users promises to be an area for ongoing refinement, so the user has responsibility of choosing the problem-solving strategy, but is not slowed down by tedious clerical work.

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