Argumentative Effects of a Geometric Construction Tutorial System in Solving Problems of Proof

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Abstract This case study aims to understand how an intelligent tutorial system, geogebraTUTOR, contributes to the students argumentative processes. Data consisted of four geometrical problems proposed to a group of students aged 16-17. Qualitative analysis of one selected case led to the identification of the development of argumentative competences by the students, as well as the level of influence produced to them. As regards the influence of geogebraTUTOR on the students, the study revealed that the interactions of tutor-teacher-student produced a significant number of mathematical learning opportunities of 'thinking strategically' type; establishing figural inference conjectures and fostering the transition from empirical to deductive argumentations.

Keywords Intelligent tutorial system; geogebraTUTOR, Mathematical learning opportunities; Argumentative competence; Geometry problem solving; Mathematical proof

Introduction

In mathematics education, most of the curricular frameworks, such as the PISA 2012 Mathematics Framework (OECD, 2013) and the Principles and Standards for School Mathematics from the National Council of Teachers of Mathematics (NCTM, 2000) establish three key aspects to consider in the teaching/learning process: (1) the acquisition of the *mathematical competence*, understood as "the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (Niss, 2003, p. 7) and within it the *reasoning and argument* capability; (2) the required *attention to diversity* both from capacities as well as the individual care, inside and outside the classroom. Last but not least, (3) the use of technology in everyday mathematics teaching.

Geometry in secondary education, due to its nature, leads to an *argumentative* process through problem solving, understanding argumentation as "the capacity to produce oral or written statements that allow conclusions to be reached, both demonstrating a proposition or persuading or convincing someone" (Planas, 2010, p. 116, our translation). The use of technological resources or *artefacts* such as dynamic geometry software (DGS) like GeoGebra can be very helpful. However, it requires an appropriate *instrumental orchestration* from the teacher (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010) to maximize *mathematical learning opportunities* (MLO). Along this line of ideas, the *intelligent tutorial systems* (ITS) in geometry are tools that support the attention to diversity and personal orchestration.

Different studies have addressed social interactions as a MLO element both in problem solving with small groups (Hitt & Kieran, 2009; Sfard & Kieran, 2001; Yackel, Cobb, & Wood, 1991) as well as in a whole class (Ferrer, Fortuny, & Morera, 2014; Morera, 2013). However, the study of individual tutor-teacher-student interactions with geometry problems is relatively unexplored. The ITS were born from the idea of having an artificial tutor accompanying the student during problem solving. Our system, called geogebraTUTOR, was made from two complementary approaches (Richard, Fortuny, Gagnon, Leduc & al., 2011). The first one aims to develop the geometric thinking of the students while, in the solving of Euclidian geometry proof problems, they write mathematical sentences (Tessier-Baillargeon, Richard, Leduc & Gagnon, 2014). The system returns discursive messages when the students are blocked in their discovery process (search for a conjecture) or in their reasoning (write of a mathematical proof). Now, the second approach, which is that we are studying in our paper, helps students in the construction of their geometrical figure (Cobo, Fortuny, Puertas, & Richard, 2007). In this context, we have considered the following research question:

In what way does geogebraTUTOR contribute to the development of argumentative competence by solving geometry problems in students aged 16-17?

Our hypothesis is that the way in which ggbTUTOR interacts with the student during the problem solving will have a certain degree of influence on the development of argumentative competence. To approach the research question, we have established an objective: to *evaluate the effect of the tutor-teacher-student interactions in the generation of argumentative MLOs*.

Theoretical Framework

As regards the research question and objectives, we have structured the theoretical framework into three parts: (1) analyse the argumentative competence and the notion of MLO, the aspects and practices that characterizes and provokes it; (2) ggbTUTOR key features and technical architecture; and (3) characterize the types of instrumental and human-artificial orchestration.

Argumentative Competence and MLO's Notion

The arguments produced by the students can range from simple observations to formal and structured arguments. Gutiérrez (2005), referring to Balacheff, identifies different types of *argumentations* classified into two groups: (1) empirical, characterized by the use of examples as the main element of conviction, differentiating three types within this group: *naïve empiricism, crucial experiment* and *generic example*; and (2) deductive, characterized by the decontextualization of the arguments used, distinguishing two more types: *thought experiment* and *formal deduction*.

The notion of MLO has been extensively studied in different studies (for example, Brewer & Stasz, 1996; Cobb & Whitenack, 1996; Cobo, 1998; Morera, 2013; Yackel et al., 1991). According to Ferrer, Fortuny, et al. (2014), they consider MLOs are those relations between the aspects of mathematical learning (conceptual and procedural) together with the artefacts used (books, blackboard, computer, etc.) and the set of actions –systematic of preparation and orchestration— that potentially will facilitate its learning.

Beyond the fact of identifying MLOs, it is the analysis of the opportunities that have triggered actual learning in the student. In other words, we need to link MLOs with achieved learning by the student. According to Boukafri, Ferrer, & Planas (2015), an MLO is transformed into actual learning when evidence exists in terms of oral or written argumentative competences by the student that denote changes or *transitions* from an *empirical* to a *deductive argumentation* in his/her mathematical knowledge. Therefore, it is reasonable to think that the MLOs will evolve alongside the systematic preparation and the *particular* and *singular* orchestration, in our case through the interactions *tutor-teacher-student*.

Intelligent Tutorial Systems

The idea of a tutorial system accompanying the student in the problem solving process is not new. The first generation ITS started in the mid-eighties, but it is not until beginning of current century –coinciding with the widespread access to IT in all areas, including education— when research in this field becomes more active.

We distinguish two groups of ITS, depending on the paradigms of reference they are based (Richard et al., 2011): (1) formal geometry, that relies on an axiomatic approach, formal and deterministic for the development of competencies and (2) cognitive geometry, where the mathematical activity occurs mostly during the geometrical shape construction, allowing a proof and refutation dialectic during the problem solving process.

Already situated in ggbTUTOR, it defines itself as an ITS that accompanies the student –complementing or replacing the teacher—in the resolution of problems with a high level of cognitive demands, by managing all the flow of discursive messages with the student. By *problems with high level of cognitive demands*, Stein & Smith (1998), it refers to those that (1) use *procedures with connections* in a way that encourages students to create connections between a network of processes and mathematical concepts; and (2) *doing mathematics*, which requires complex and non-algorithmic thinking, selfregulation and access to relevant knowledge and previous experience to make appropriate use in working through the resolution of the problem.

Technical architecture

ggbTUTOR is a portable, web-based application, which is accessible from any Java enabled Internet browser. Its architecture is composed of three main components: (1) the ggbTUTOR interface, (2) the tutor subsystem and (3) the database, with the following characteristics:

1. The ggbTUTOR interface (Fig. 1), which provides the student all tools needed to solve a geometrical problem, it supports the reasoning process through figural inferences in the approach designed by Duval (1995) and Richard (2004). The ggbTUTOR interface is divided into the following areas: graphical construction (GeoGebra module), where the student creates the graph propositions (parallel, perpendicular lines, etc.); deductive area (deductions & justifications), where the student writes the argumentative propositions; and activity log, which includes the graphic and deductive record as well as the tutor messages.

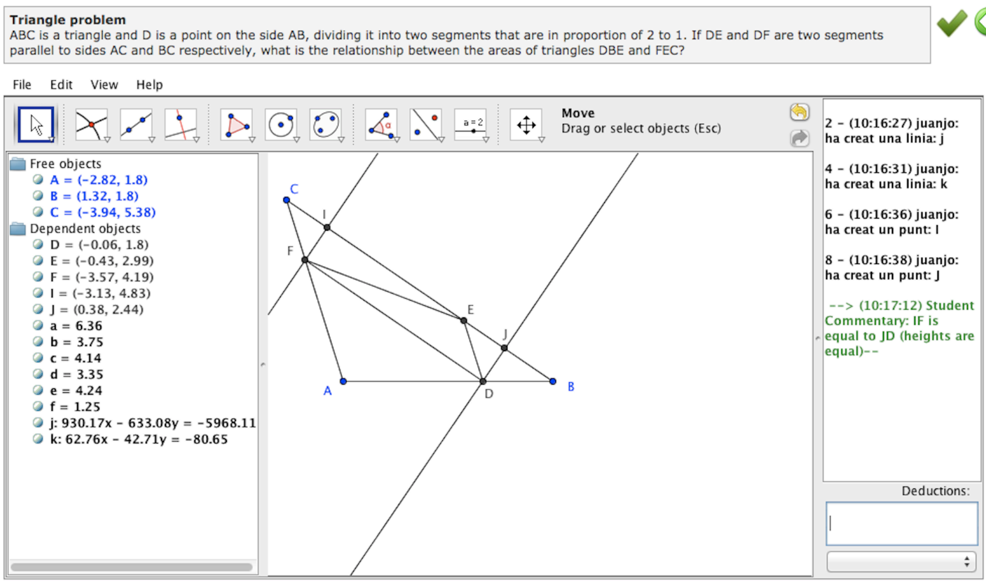

Fig. 1 GeogebraTUTOR interface

2. Tutor subsystem, which has as its mission to supervise and help the student along the problem solving process. It consists of two modules: agent mediator, which receives and processes all the graphical and deductive prepositions; and agent tutor, which is responsible for displaying certain messages to the student, according to the message selection algorithm (Fig. 2).

3. The database, located at the ggbTUTOR server, stores all the data of problems, users, etc. Therefore, ggbTUTOR requires two main elements: (1) the problem solution tree, built beforehand by the teacher, which is based on the basic space of the problem (Cobo, 1998); and (2) the message selection algorithm. Both pieces provide the system with the ability to interact with the student during the problem solving process.

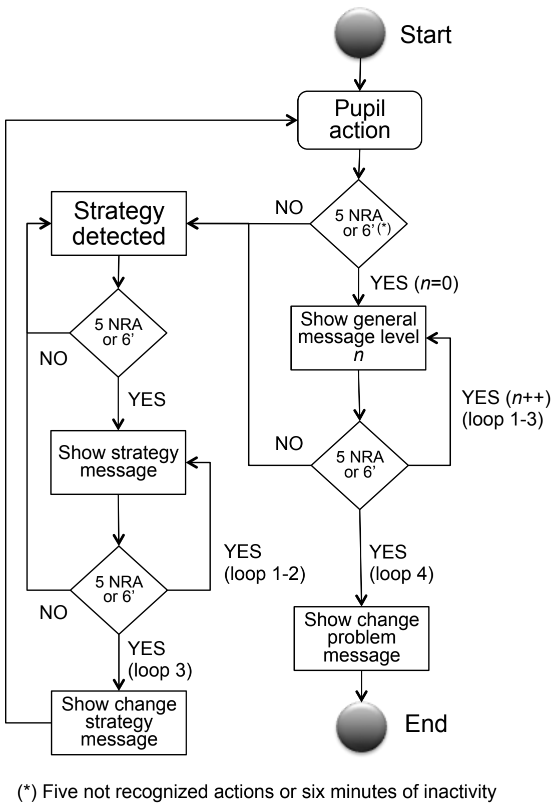

Fig. 2 Agent tutor message selection algorithm

In this way, when the student starts a problem, ggbTUTOR is listening to his/her activity, so when he/she performs an action that is part of one of the preloaded strategies, ggbTUTOR detects it accordingly. Furthermore, if the student makes either five invalid actions or does not take any action during six consecutive minutes, ggbTUTOR sends a message (Fig. 2). In this case, we can say that an interaction tutor-teacher-student has occurred, which will have certain influence on the problem solving process and eventually resulting in a MLO.

Instrumental Characterization and Human-Artificial Orchestration

The systematic of preparation and orchestration based on certain resources and techniques is the key to promoting productive episodes. For this reason, the way in which the teacher analyses in advance all the aspects of the proposed problems in ggbTUTOR that will be later carried out during the orchestration of the tutor-teacher-student interactions, it follows a four phase systematic adapted from Morera (2013) and Smith & Stein (2011) : (1) anticipation, (2) expanded didactical configuration, (3) mode of operation, and (4) monitoring.

The anticipation phase (1) consists of making a prior analysis on how the students can approach the problem, and anticipate their possible answers. This involves preparing a detailed study of all the possible ways to solve it, which possible messages may ggbTUTOR handle and when it would show them to the student, in case he/she encounters difficulties during the solving process. A key instrument during this phase is the basic space of the problem (Cobo, 1998) consisting of a tree structure, in which its branches show the different strategies that a student could follow to solve the problem. For the wrong strategies, different tutor messages are included, with the objective of guiding the student during the solving process, providing only the minimum help, looking for the right balance between helping the student too much and leaving him/her blocked for too long. The expanded didactical configuration (2) describes the set of artefacts that the teacher decides to include in the didactical unit. The mode of operation (3) refers to the way in which the teacher interprets a didactical setup to meet his/her didactical intentions. Finally, the monitoring (4) is based on the follow-up of the mathematical thinking and the resolution strategies from the students while they are working on the problem.

Method

In line with other studies on MLOs and student interactions, such as (Cobo et al., 2007; Doorman, Drijvers, & Gravemeijer, 2012; Ferrer et al., 2014; Morera, 2013), we will follow an approach to qualitative methodology in which we will make a case study. In particular, we will look to learn from the tutor-teacher-student interactions and find evidence of connections between argumentative MLOs to obtain a better understanding of the transition from an empirical to a deductive argumentation in a technological learning environment.

Design and Data

The case study is based on a didactical sequence of Geometry where, over two sessions, we proposed four tasks to a group of 16-17 year old students. For this study, we selected the cases of two students, Laura and Oriol. The chosen students had a good academic performance in mathematics, were interested in technologies and had good communicative skills. This last aspect was especially important, as we were interested in working with students capable of expressing the argumentation process full of richness.

The students had little previous experience in use of GeoGebra and DGS in general, so prior to the experiment they received a brief GeoGebra training. It is also worth mentioning that students were not familiar in practicing argumentative competence in the daily classroom activities. Instead, the work methodology they used in class combined the expositive sessions by the teacher with a variety of theory application activities, where students worked in pairs or groups of three with a large degree of freedom to comment on the activities, but they rarely worked individually.

In addition to the students, a key participant in the experiment was ggbTUTOR –the artificial tutor—as well as a teacher –the human tutor— that eventually could interact with the students through messages with certain mathematical content.

The four selected tasks are geometrical problems that compare areas of plane figures. Several reasons encouraged us to use this typology of problems: firstly, because they are tasks with a *high level of cognitive demands* in the sense described in the theoretical framework. Secondly, because they have been tested in multiple studies (Cobo et al., 2007; Cobo & Fortuny, 2000; Cobo, 2004; Richard et al., 2011; Richard, Iranzo, Fortuny, & Puertas, 2009). Finally, because they are suitable problems for students aged from 14 onwards, with an adequate complexity level to be considered by them as 'problems', but without being unsolvable. The design of the problems in ggbTUTOR –in line with the systematic of preparation and orchestration— requires an in depth analysis of all the mathematical aspects and possible ways to solve them. To do this, we will use the basic space of the problem (Cobo, 1998).

For this study, we selected two problems, a *triangle problem* and a *quadrilateral problem* (see Fig. 3 and 4).

Triangle problem

ABC is a triangle and D is a point on the side AB, dividing it into two segments that are in a ratio of 2 to 1. If DE and DF are two segments parallel to sides AC and BC respectively, what is the relationship between the areas of triangles DBE and FEC?

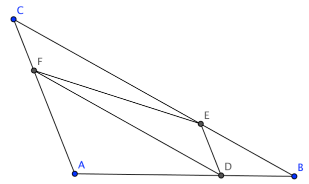

Fig. 3 Statement of the triangle problem

Quadrilateral problem

ABC is a triangle and E and F are the midpoints of sides BC and AC, respectively. If D is any point on the side AB, what is the relationship between the area of DECF quadrilateral and the sum of areas of DBE and ADF triangles?

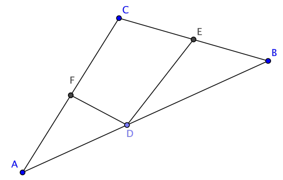

Fig. 4 Statement of the quadrilateral problem

The triangle problem can be solved through three main strategies: use of Thales's theorem, decomposition into unit triangles, and identifying similarity (Fig. 5), while the quadrilateral problem can be solved by using four different strategies: (1) dividing DECF quadrilateral by DC diagonal, then realizing that the resulting triangles share base length and D vertex with DBE and ADF triangles; (2) dividing the quadrilateral by the FE diagonal, then noticing that the shared base of resulting triangles is parallel to AB side; (3) drawing parallel lines to midpoints; (4) and with direct application of formulas.

To summarize we can say that both problems have equal difficulty and use similar mathematical concepts and procedures, of which we highlight: proportionality, parallelism, perpendicularity; similarity, equivalence and congruence of triangles; use of Tales' theorem, figure decomposition and search of particular cases.

Under these conditions, each of the students participating in the experiment was asked to solve the four problems with the eventual help of the human and the artificial tutor. We transcribed all the data of the solving process for each of the problems through the recordings of the ggbTUTOR log, screen capture and audio. The resulting transcriptions were characterized and analysed from a qualitative perspective to investigate our research question.

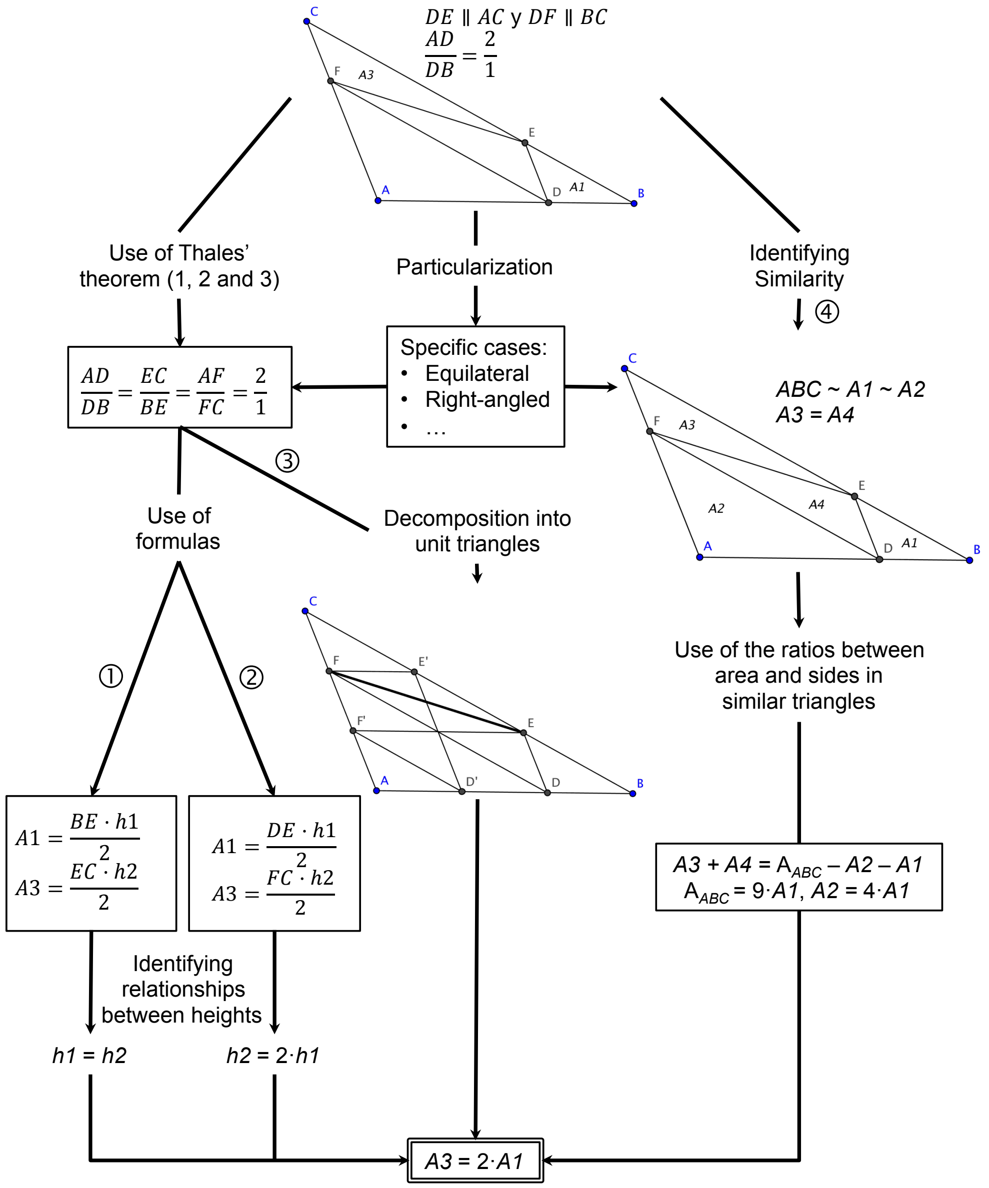

Fig. 5 Basic space of the triangle problem

Analysis

To analyse the transcriptions of the student problem solving process, we designed an instrument to enable us to characterize the tutor-teacher-student interactions, together with the different argumentative MLOs that could arise.

Firstly, we looked at the instrumental side of the tutor-teacher-student interactions, in order to understand how and in which way the artefacts were used along the didactical sequence. To achieve this we have relied on the six orchestration types from Drijvers et al (2010): *Technical-demo, Explain-the-screen, Link-screen-board, Discuss the-screen, Spot-and-show, Sherpa-at-work.*

The first three orchestration types are dominated by the tutor-teacher actions, while students dominate more in the last three types.

Secondly, we characterized the tutor messages to analyse the type and level of influence on the student. To achieve this, we looked at two dimensions: (1) the type of message –the cognitive dimension— according to the communicative processes that emerged between tutor and student during the problem solving (Cobo & Fortuny, 2007): *conceptual, heuristic, metacognitive* and *semiotic*; and (2) the level of message, depending on the level of information they contained (Cobo et al., 2007; Cobo & Fortuny, 2007): *level 0* were general messages without mathematical content, *level 1*, which contained little relevant information, and *level 2* that contained more detailed information, but without being complete to non-transform the problem to a merely simple activity.

Finally, we characterized the types of MLO according to the main groups defined by Morera, Planas, & Fortuny (2013): *mathematical contents, thinking strategies,* and *self-regulating activities*. As we were especially interested in the argumentative processes, we looked in more detail on at those MLOs within the *thinking strategies* group, together with the type of argumentation produced by the student, by using the classification made by Gutiérrez (2005), as detailed in the theoretical framework.

To analyse the problem solving process, we divided the transcript into episodes, which are time periods where the student completes a phase of the process followed, as defined by Cobo & Fortuny (2000).

Results and Discussion

The results of the study are presented in the form of a storyline of the student learning process. This storyline is illustrated with two examples of student work and is empirically supported by qualitative findings.

During the first session and prior to the starting with the first problem, the teacher introduced ggbTUTOR to the students through a *technical-demo* orchestration (Drijvers et al., 2010), where he highlighted the main differences with GeoGebra, stressing the fact that ggbTUTOR is continuously listening to the mathematical student thinking, in the sense of Leatham, Peterson, Stockero, & Zoest (2015), and based on the fact that it makes decisions on whether to proceed or not with the guided resolution, by showing different types of messages.

Additionally, as the students were not familiar in practicing the argumentative competence in the daily classroom activities, the teacher placed special emphasis on this subject by reviewing the meaning of argumentation in mathematical problem solving and by showing examples of what is a valid argument in secondary education. For example, the teacher presented examples to the students to illustrate what is a valid argument in secondary education (i.e. "*due to being perpendicular they form a 90º angle*" is a valid argument, versus "*because it can be clearly seen in the figure*" that it is not valid).

Thinking Strategies Analysis

The results of the initial open-ended group activities of the first teaching experiment show a variety of strategic lines and conjectures. When students became familiar with the problem, they started looking for relationships between the elements of the geometrical figure, starting by comparing angles and writing down their thinking in the ggbTUTOR deductions area (for example the case of Laura in the action 6):

6. Teacher: At a first glance I don't see any relationship between the angles.

Even if it is not very common that students start looking at angles, and although her reasoning was not on the right path, we consider that Laura initiated the work of the argumentative competence. However, before Laura confirmed the observation she made, ggbTUTOR sent her a *metacognitive* message of *level 0*:

7. ggbTUTOR: *The concepts associated with the figure or the problem statement suggests any new information?*

The message influenced Laura by the fact that she stopped her writing, re-read the problem statement and meditated about the search of possible ways to approach the problem in order to establish the strategic line to try to solve it. For this reason, we consider that the ggbTUTOR message had an influence on the student, producing a MLO of *thinking strategies* type, acting both as the start and generator of the whole episode.

After the ggbTUTOR message, Laura established a *deductive conjecture* (actions 8 and 9).

8. Laura: *Student Commentary: Taking into account that the AB segment is divided into a ratio of 2 to 1; I have the impression that the parallels to AC and AB have divided these segments into the same proportion. To prove this I will use the Thales's theorem.*

9. Laura: *Student has added an Argument: Thales's theorem.*

Laura formalized the mathematical procedures that would be used to prove it, by using previous knowledge (Thales's theorem). Laura tried to divide the BC segment into three equal parts by using the Thales's theorem; however she did not follow the process properly, as she drew an auxiliary segment CI, but started the process of drawing auxiliary circumferences from point I, which is just the other way around.

This is a common mistake that many students make when dividing segments into equal parts. In any case, Laura divided the BC segment properly, as she managed it by drawing the auxiliary segment with a known length (Fig. 6).

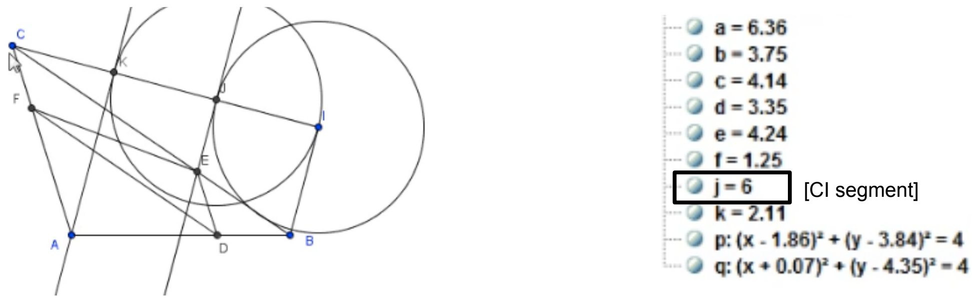

Fig. 6 Auxiliary lines drawn by Laura during the episode of analysis and execution

At this point, ggbTUTOR recognized the graphical actions made by Laura previously and established that she was following the strategy "*use of Thales's theorem*" (Fig. 5).

Finally, Laura argued that the BC side was following the same proportion rules as AB, even though it said incorrectly a ratio of 2 to 1, when it should be 3 to 1 when referring to BC, and concluded –without proving— that the same should happen with AC (action 24).

24. Laura: *Student Commentary: Given that it coincides in the BC segment, which is in a ratio of 2 to 1, I can conclude that the three segments are in the same ratio.*

From the analysis of the episode, we can conclude that it commenced with the beginning of the argumentative competence by Laura, which was influenced by ggbTUTOR by producing a MLO of *thinking strategically* type, in which Laura meditated and established a strategic line to solve the problem, which was to find the ratio between the sides of the triangles. She started with the sides shared with the main triangle (DB, BE from DBE and EC, CF from FEC) by establishing a deductive conjecture and demonstrating it in actions 8 and 24, respectively.

Once Laura proved the relationship of the shared sides with the ABC triangle, she deleted all the auxiliary elements that she used in the previous demonstration. This fact coincided with a *metacognitive* message of *level 0* that the teacher sent to all the students in which he gave them an advice that they would probably need drawing auxiliary lines to solve the problem. This message made Laura think about it, asking a question back to the teacher for advice on her initial decision to remove the auxiliary drawings (see actions 42, 43 and 44).

42. Teacher: *To solve a geometry problem, the drawing of auxiliary lines to the given figure is usually needed, mainly to justify it.*
43. Laura: *So, do we have to leave the auxiliary drawings?*
44. Teacher: *Sure, yes, yes. Well as you prefer. As it is all recorded, we will see it anyway*

For this reason, we consider that the teacher's message had an influence on Laura, by making her meditate on the fact that she possibly took a wrong decision, and by understanding that keeping auxiliary elements used in the argumentation is key in geometrical demonstrations. For all of that, we consider that the teacher message led to a MLO of *thinking strategically* that would potentially be leveraged in subsequent episodes.

Continuing with the line strategy of finding the relationship between the triangle sides, Laura tried to find the relationship between the remaining sides of DBE and FEC triangles, which are the DE and FE sides, respectively. To do this, Laura drew three auxiliary circumferences, together with three segments, in order to compare the length of the DE, CF and CE, FE sides, respectively.

Laura made the previous comparison empirically, instead of noticing that FDEC formed a parallelogram (Fig. 7), and she then argued her findings through an empirical argumentation (action 57), and therefore weakly.

57. Laura: *Student Commentary: By drawing two circumferences, I have been able to check, as it looked at first glance, that the CF and DE sides are equal*

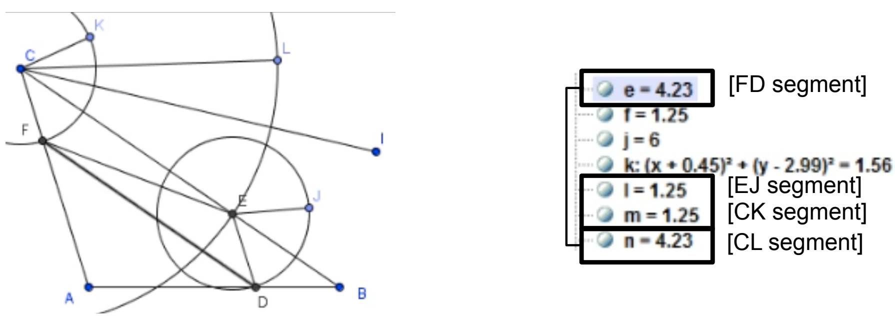

Fig. 7 Auxiliary lines drawn by Laura

We have noted that in the arguments produced by Laura, that she did not make any explicit relationships: "*in a ratio of 1 [or 2] as regards their respective original segments*" (actions 58 and 62), hence we have interpreted that she referred to ratios of 1 to 3 and 2 to 3, respectively.

58. Laura: *Student Commentary: In addition to this, I know that the three segments that form the DBE triangle are in a ratio of 1 in respect to their respective original segments (DE from AC, BE from BC and DB from AB)*

… … …

62. Laura: *Student Commentary: While FEC triangle is formed by one [side] in ratio of 1 (CF) and the others (CE and FE) are in ratio of 2 in respect to their originals*

Also, Laura made a mistake in stating that FE side would also follow a 2 to 3 ratio with its "*original*", while in this case we cannot be sure which segment she was pointing out, we suspect that she was referring –incorrectly— to the BC side.

From the analysis of the episode, we conclude that Laura completed the finding of the relationship between the sides of DBE, FEC and ABC triangles through an empirical argumentation with two main inaccuracies: she did not make the relationships explicit and made a mistake with the relationship of the FE side.

Figural Inference

Laura tried guessing the possible relationship between the area of triangles, by establishing a figural inference argumentation and visualizing in her mind the possible result without making it public. We interpret that Laura visualized the proportions of the heights of the triangles. The teacher reminded her that she had to argue everything appropriately, so Laura explained to him the strategy she was following, which he finally validated (actions 74 to 77).

74. Laura: *If I have some relationships because I have done it graphically, but I have done the proportions mentally?*
75. Teacher: *Ok, but you will have to explain it. You have to justify it*
76. Laura: *So, what should I do? As I have the proportions of the sides, then should I put the relationship?*
77. Teacher: *Well, that's it, but you have to write it…*

Laura started looking at the heights of the triangles, with the idea of using the formula for the area of the triangle to obtain the final result, once she had found the required relationship. Based on that, Laura drew the heights of DBE and FEC triangles from E vertex to DB and FC sides, respectively, to compare the measurements (Fig 8).

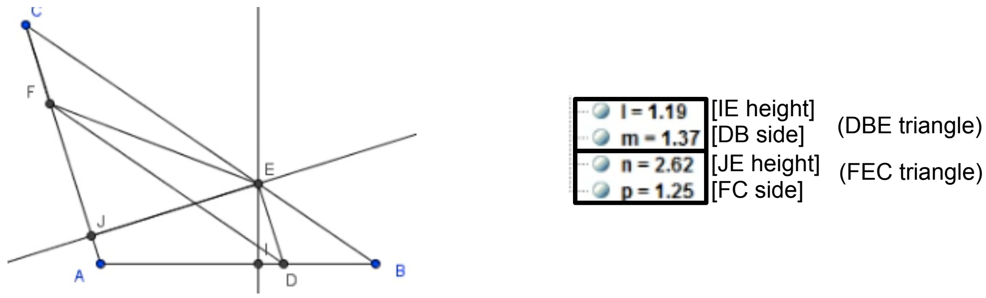

Fig. 8 Auxiliary lines drawn by Laura during episode 5 of execution

However, she did not choose the appropriate heights, as she should have opted for those on the parallel sides. In addition, instead of making the comparison between the heights, Laura compared empirically lengths of heights and sides of the triangles; hence she did not reach any satisfactory conclusion.

From the analysis of the episode, we conclude that Laura started with a mental visualization of the possible result by establishing a figural inference conjecture, as she suspected that the heights of the triangles could follow similar proportion rules as the corresponding triangle sides. To prove her conjecture, she started drawing the heights of the requested triangles to check the measurements, but she made the wrong decision twice: drawing the heights on the non-parallel sides, and comparing the lengths of heights and sides. Laura failed in her strategy, so she had to think of a new plan in subsequent episodes.

From Empirical to Deductive Argumentation

Laura abandoned the strategy line she followed during the previous episode, by deleting all the auxiliary lines that she had drawn. Being observed by the teacher, Laura was asked by him with a *contextual level 0* message: "*Have you finished?*" (action 93), while she answered: "*No, because I have realized that formula for the area is not this one…*" (action 94), referring to the impossibility of making the comparison that she tried before.

After discarding the previous strategy, Laura wrote an argumentation: "*Given that ED side is parallel to AC, I can conclude that the DBE triangle is three times smaller than ABC (…). With the same respect, we can say that the FEC triangle has a ratio of 5/9 with ABC*" (actions 95 and 96), trying to establish a relationship between the area of the DBE, FEC triangles and ABC.

From the analysis of the episode, we conclude that Laura established a *deductive argumentation*, in which she made a qualitative step forward –from relationships between sides of triangles to relationships between areas of triangles—even though with few considerations: firstly, the argument that Laura made was wrong due to thinking that the area ratio would be also be linear. This is a common error in secondary school students (Hart, Brown, & Kuchemann, 1981). Secondly, dragging the mistake that Laura made earlier with the ratio of the FE side of the FEC triangle, she established a ratio of 5/9 that we did not manage to interpret, although we presume that she came to it by making a kind of sum with the proportions of the sides of the triangles.

Once Laura found the relationship between the area of the DBE, FEC triangles and ABC, she had to make the last step to obtain the final problem result. While she was thinking how to approach this, ggbTUTOR sent a *contextual* message of *level 1* (see action 97) that gave Laura an important hint that made her think, and thereafter to start a brief discussion with the teacher (see actions 98 to 104).

97.	ggbTUTOR:	*Remember the number of heights of a triangle and how to draw them*
98.	Laura:	*What does this mean [pointing out the ggbTUTOR message]? It's giving me a hint?*
99.	Teacher:	*Yes. It is a message that the tutor is sending you.*
100.	Laura:	*I think I know the proportion, but I don't know [how to prove it]*
101.	Teacher:	*Then write it down. That's a conjecture.*
102.	Laura:	*I know the ratios of the sides of the triangles, all of them. So I think I know the [triangle area] proportion*
103.	Laura:	*But I don't know the relationship between these two [pointing out DBE and FEC triangles]*
104.	Teacher:	*This is the problem. Write down everything you know about.*

Laura conjectured through a *mental visualization* what could be the relationship between the area of the DBE and FEC triangles through the ratios of its sides with the ABC triangle, however she did not know how to make the final step.

The message from ggbTUTOR was very appropriate, as Laura did not choose the right heights before. For this reason, we consider that the ggbTUTOR message led to an MLO of *thinking strategically,* with immediate effect on Laura, as she got back to the strategy that she previously abandoned and drew the three heights of DBE and FEC triangles (Fig. 9).

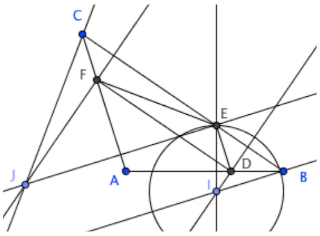

Fig. 9 Auxiliary lines drawn by Laura

However, Laura kept comparing empirically, and incorrectly, heights and sides of the same triangles instead of looking at the heights of the two triangles. After few attempts to elucidate the remaining geometrical connection, she asked the teacher to try to find out more about how she could obtain the required relationship. Finally, the time allowed to solve the problem was over and Laura did not get to complete the resolution.

From the analysis of the episode, we conclude that the ggbTUTOR message sent to Laura influenced completely in how the episode was developed, as established a continuity in the solving process, linking with the strategy line that Laura discarded previously and triggering a MLO of *thinking strategically* with immediate effect to her, which caused a continuity in the solving process and drawing the three heights of the two triangles. However Laura didn't change the comparison elements and kept looking at the heights and respective sides of the corresponding triangles, so she did not manage to solve the problem completely.

Tutor-Teacher-Student Interactions

In terms of tutor-teacher-student interactions, there were several key moments that produced MLOs of *thinking strategically*: firstly, the one produced by the teacher when he made Laura meditate and internalize about what type of elements were required to produce valid demonstrations (actions 42 to 44). Secondly, another key moment produced by ggbTUTOR that made Laura return to her initial plan (actions 97 to 104).

Finally, to highlight a moment propitiated by the teacher when he explained the 'dragging' functionality of DGS artefacts (Gutiérrez, 2005), before the start of the quadrilateral problem (Fig. 4), which influenced completely the development of the whole problem solving process. This is the case of Oriol, a student, who –using his geometrical view— commenced the quadrilateral problem by dragging the figure and by visualizing a triangle with a regular shape, so he based his strategy on that. Oriol achieved a particularization *empirically* (equilateral triangle, Fig. 10).

Fig. 10 Figure dragging and auxiliary lines made by Oriol

Oriol obtained the problem answer directly, however he based his proof on a misconception of the demonstration in dynamic geometry, as he thought –wrongly— that if he was able to find the answer in a specific particularization, the relationship would remain when dragging the figure back to its initial shape.

Later, during Oriol's search for the path to the generalization, he came back to his initial idea, by asking the teacher about the validity of his misconception of the demonstration in dynamic geometry (action 32).

29.	Oriol:	*Student Commentary: From this point, independently, as we move [drag] the triangle, the base of the CEF triangle is equal to the base of the EDB triangle, and the CF side of the CEF triangle is equal to the AF side of the FGA triangle*
…	…	...
32.	Oriol:	*If we say that due to the fact that the problem asks us for the relationship between the quadrilateral and the sum of triangles and we state that it is always the same as I found it for one case, then it will be the same for all… It's enough, or do I have to justify more?*
33.	Teacher:	*The reason that you have given is met in all cases? (…) If you are able to prove that it keeps fulfilling, then it's fine, otherwise not (…) Why have you noticed that it worked?*
34.	Oriol:	*Well, because I have used a very particular case where all the triangles are equal*
35.	Teacher:	*And this is still met?*
36.	Oriol:	*Yes*
37.	Teacher:	*And why it fulfils? That's what you have to think about*

The teacher replied to Oriol with a *metacognitive* message of *level 1*, trying to explain to him that he needed to provide a more deductive kind of demonstration to be an acceptable answer (action 33). Oriol argued that the condition should be met for all cases with a conjecture of *crucial experiment* type, as it was based on his carefully selected example (equilateral triangle) and his misconception (actions 34 and 36). The teacher insisted that he had an argument provided was not enough, as he had to find out why the condition was always met. Given this fact, Oriol – during few minutes— tried to think about alternative ways to be provided with a complete demonstration. Finally, the time allowed to solve the problem was over, and Oriol did not manage to prove the general case.

Oriol tried to make a qualitative step forward to the generalization, but kept providing empirical argumentations and maintained in his mind the misconception of demonstration in GDS environments. As stated by Gutiérrez (2005) "it is an obstacle for the students to understand the need of the deductive demonstration and to learn how to produce this type of demonstration (…) in other studies, when getting to this blocking point, the teacher has introduced the deductive demonstrations to the students in a way as to understand why the conjectures are true" (p. 43, our translation).

Conclusions

In this study we proposed a research question with the main aim of evaluating the effect of the tutor-teacher-student interactions in the generation of the argumentative MLOs. The analysis of the results of the cases of Laura and Oriol has revealed that the development of the argumentative competences they produced and the level of influence of ggbTUTOR to the students.

Regarding the development of the argumentative competence, we have noticed a trend, both from Laura and Oriol, in the use of empirical argumentative competences, although Laura always tried to think deductively, she often relied on empirical data. This is a known effect on DGS environments (Gutiérrez, 2005) that we also experienced with ggbTUTOR. In terms of the quality of the arguments produced, we have seen a positive change on the evolution of Laura's and Oriol's progress of argumentative competences during the problem solving process, and in two main aspects: in the lexical and semantic quality and in the qualitative, with a step forward from particularization to generalization.

Regarding the influence of ggbTUTOR on the students, the study revealed that the tutor produced a significant number of MLOs of *thinking strategically type*, establishing figural inference conjectures and fostering the transition from empirical to deductive argumentations, which had a positive effect on both Laura and Oriol, as it guided them through the path of the problem resolution. This is undoubtedly a clear advantage of the ITS artefacts that considerably help in the attention to diversity, since unlike the teacher; they are able to continuously listen to the student thinking and, based on that, they can take the appropriate decision, sending different types of messages to the student.

The study also revealed some negative influences of ggbTUTOR, which is a legacy from the DGS environments: the misconception of demonstration, and the obstacle in understanding the need of the deductive demonstrations in dynamic geometry. This latter aspect has also been discussed by Gutiérrez (2005). These two obstacles should be considered when designing didactical sequences that use DGS artefacts.

Finally, we consider that the study of the ggbTUTOR has contributed to the exploitation of the argumentative MLOs and how the development of the argumentative competence evolves. This deserves further investigation that could be continued by introducing didactical sequences with problem itineraries, followed by whole class discussions and by studying the ggbTUTOR interactions with students working in pairs.

ggbTUTOR poses a simulated didactical relationship in which the intelligent tutor plays, in spite of a personalized attention according to the iterative learner's model, a teacher role that is complementary to the role of the regular teacher that only accompanies the student. Even if the teacher does not teach as such, he supports the student in the review and return of the problems and, although indirectly, institutionalizes certain fragments of knowledge. The teacher acts as a peer with ggbTUTOR, so even if he often interacts with the student and impacts his milieu, the teacher's actions and interactions remain secondary. In the same way, ggbTUTOR can send certain messages to the student, but these remain subordinate to the interactions, raising mathematical knowledge.

ggbTUTOR provides each student with a different experience by offering a just-in-time feedback oriented towards helping the student development argumentative competences.

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